Part 8: Math Section Training and Walkthroughs

In part 7, we just saw how to attack the SAT Reading section by relying on literal reading and objectivity, and by remembering the important differences between the SAT and a normal test or discussion in a high school setting. Now, we'll build on this foundation in order to address the SAT Math sections. You'll see how we combine our awareness of the test's unique design with our existing math knowledge and our reading skills, allowing us to find correct answers quickly and efficiently.

In this part, you'll learn the following:

- why the SAT Math sections are unlike the math tests you take in high school
- why the SAT Math sections had to be designed in a particular way to make their results useful for colleges
- the single biggest secret of the design of the SAT Math sections
- the two critical components of success on the SAT Math sections
- all of the basic math ideas that you'll need on test day
- the two key ideas related to statistical sampling that the College Board allows itself to test
- the ways that variables, coefficients, exponents, and constants can affect the output of a function
- what backsolving is, why it helps some test-takers a lot, and how to apply it in ways most people aren't aware of
- why it's so important to consider every answer choice on the SAT Math section—and how it can save you time
- the uses of rounding and estimation that aren't obvious to most untrained test-takers
- the unwritten rules of the SAT Math sections
- why formulas matter much less on test day than most people would expect
- why every real SAT Math question can potentially be answered in under 30 seconds
- the hidden patterns of SAT Math sections, and how they can help you attack and check questions efficiently
- the 3 major types of approaches to an SAT Math question, along with the advantages and drawbacks of each
- the recommended 7-step "Math Path" for attacking questions quickly and effectively on test day
- how the SAT's provided diagrams can sometimes be used to answer questions with total certainty at a glance
- how to deal with "grid-in" questions
- how to work through Roman numeral questions
- why it can be important not to think about "showing your work" on test day
- why the "order of difficulty" really doesn't exist
- how to apply these concepts to every math question from the first four Official SAT Practice Tests.
- and more . . .

SAT Math Training

> *The essence of mathematics is not to make simple things complicated, but to make complicated things simple.*
> S. Gudder

Overview and Important Reminders for SAT Math

The Math questions on the SAT are a very mixed bag. The current version of the SAT features several different types of math; almost everything you could study in high school math is on there except calculus, advanced trig, and advanced statistics. On top of that, an individual question can combine concepts from any of those areas, which often makes the questions hard to classify.

Some test-takers cover all the basics of SAT Math before they reach high school, and some take geometry as seniors and never even have classes in algebra. For the first type of person, SAT Math concepts are almost forgotten; for the second type, they're just barely familiar.

In short, nobody I've ever met has felt completely comfortable with all the math on the SAT when they began training, for a variety of reasons. Don't let it bother you!

But that's not all—mastering the key mathematical concepts that can appear on the SAT still won't guarantee a high score. In fact, you probably know some people who are "math geniuses" who still don't make perfect scores on the SAT Math section. You might even be one of those people yourself.

For those people—and for most test-takers, actually—there's something missing when it comes to SAT Math. There's a key idea that they haven't realized yet.

What idea is that? It's the fact that the SAT Math test isn't primarily a math test . . . at least, not in the sense that you're probably used to. You need more than just mathematical knowledge to do well on the SAT Math Section. Think of it as a bunch of problem-solving exercises. Actually, the better you get at SAT Math, the more you'll come to realize it's just a game—and the more you come to see it as a game, the better you'll get at it.

The truth is that SAT Math is primarily a test of your knowledge and application of mathematical definitions and properties, and your ability to identify patterns and "shortcuts" that most untrained test-takers won't be looking for. The calculations themselves generally aren't complicated—even on so-called "hard" questions—as you'll see when we go through some real test questions. The main thing that really makes SAT Math questions difficult is figuring out what they're asking you to do in the first place.

So trained test-takers do better on "SAT Math" partly because they focus on looking for the most efficient ways to set problems up, rather than automatically relying on formulas. Unfortunately, most test-takers never realize how different SAT Math is from school math, so they spend too much time trying to find complicated solutions to the problems on the SAT, as though the SAT were like a regular math test in high school. This is very frustrating, and results in lower scores. It's like trying to cook a soufflé with a hammer.

Studying this Black Book will help you use the techniques that trained test-takers use to score well on SAT Math. More importantly, you'll come to see the SAT "Math" test for what it really is: a reading and problem-solving test that happens to involve numbers!

The Big Secret of SAT Math

Before we go any further, it's important that you be in the right frame of mind when you approach SAT Math questions. As I've mentioned a couple of times so far, most SAT Math questions aren't really "math" questions at all, at least not in the way you probably think of math questions. You need to understand why this is.

Put yourself in the College Board's position for a moment. If you're the College Board, your goal is to provide colleges and universities with useful, reliable data on their applicants' abilities. It wouldn't really make sense to have those applicants take a traditional test of advanced math, for two reasons:

- Not all applicants will have taken the same math classes, so a traditional test wouldn't be able to distinguish students who had never had a chance to learn a certain type of math from students who had learned it and were bad at it.
- More importantly, the high school transcript already does a pretty good job of indicating a student's ability to answer traditional math questions.

So a traditional test of advanced math wouldn't let the College Board provide very useful data to colleges and universities. And it wouldn't make any sense to come up with a traditional test of *basic* math, either, because far too many test-takers would do very well on that, and the results would be largely meaningless.

The College Board's solution to this problem is actually kind of clever. They make sure that SAT Math questions only cover relatively basic math topics, but they cover those topics in non-traditional ways. In this way, the College Board can be fairly certain that every test-taker has the potential to answer every question correctly—but only by thinking creatively, which keeps the results of the test interesting for colleges and universities.

In fact, let me say that last part again, in all caps, and centered, because it's super important:

SAT MATH QUESTIONS TEST RELATIVELY BASIC MATH IDEAS IN STRANGE WAYS.

That idea is the thing that most test-takers don't realize. It's the thing that causes so many people to spend so much time practicing math for the SAT with so little result. The way to get better at SAT Math isn't to learn advanced math, because most SAT Math isn't very advanced. The way to get better is to learn to take apart SAT Math questions so you can understand which basic ideas are involved in each question.

For this reason, you'll often find that the most challenging SAT Math questions can't be solved with any of the formulas you normally use in math class. In general, SAT Math questions avoid formal solutions. If anything, you might even say that answering SAT Math questions is kind of a creative process, because we never know exactly what the next question will involve, even though we can know the general rules and principles underlying its design.

The Two Critical Components of SAT Math Success

Since the SAT Math section is all about basic math ideas presented in strange ways, there are two key areas of knowledge we'll need to do well on the test:

- Basic knowledge of arithmetic, geometry, trigonometry, and algebra (including some basic graph-related ideas), and
- a thorough understanding of the SAT's unwritten rules, patterns, and quirks.

So you will need *some* math knowledge, of course, but you won't need anything like calculus or advanced trig or stats, and you won't have to memorize tons of formulas. Like I keep saying (and will continue to say), it's much more important to focus on how the test is designed than to try to memorize formulas.

In a moment we'll go through the Math Toolbox, which is a list of math concepts that the SAT is allowed to incorporate when it makes up questions. After that, we'll get into the SAT's unwritten rules of math question design.

SAT Math Toolbox

In a moment, we'll talk about how to attack the SAT Math section from a strategic perspective. But first, it's important to make sure we know all the mathematical concepts the SAT is allowed to test (don't worry, there aren't that many of them).

This concept review is designed to be as quick and painless as possible—our goal here isn't to learn all of these ideas from scratch, but to review them on the assumption that you've already learned most of them in a classroom setting at some point. If you feel that you'd like a little more of an explanation for a certain topic, the best thing to do is find somebody who's good at math (a teacher, parent, or friend) and ask them to spend a little time explaining any problem areas to you.

The ideas in this Toolbox might seem easier to you than the actual SAT Math section. That's because the difficulty in SAT Math usually comes from the setup of each problem, not from the concepts that the problem involves. The concepts in this review are the same concepts you'll encounter in your practice and on the real test, but the real test often makes questions look harder than they really are by combining and disguising the underlying concepts in the questions.

For SAT Math, it's not that important to have a *thorough* understanding of the underlying concepts. All you need is a quick, general familiarity with a few relatively basic ideas. So that's all we'll spend time on.

Please note that this list is similar in some ways to lists of math concepts provided by the College Board, but my list is organized a little differently and presents the material in more discrete units. In addition, my list explains things in plainer language, omits some concepts that are redundant in College Board sources, and makes fewer assumptions about what you already know, making it easier to study. (This is also a good time to point out that the College Board provides some math questions that *don't* appear in official full-length SATs—instead, they appear in practice materials that are meant to sharpen your general math skills. These College Board math questions that appear outside of an official SAT Practice Test are often significantly more advanced and more challenging than real SAT Math questions will be, and I recommend you avoid them. The College Board probably provides these additional practice questions in an attempt to make untrained test-takers believe that the SAT Math section covers more challenging concepts than it really does. So stick to the SAT Math questions you encounter in official SAT Practice Tests from the College Board!)

As you're going through this list, you may see concepts that aren't familiar. Before you let yourself get confused, make sure you've read this list through TWICE. You'll probably find that a lot of your confusion clears itself up on the second reading. You may also see concepts that seem very familiar, basic, and boring. I would still recommend that you read through the whole list TWICE. A lot of the critical subject matter on the SAT Math *is* pretty basic and boring, and it's helpful to review that material so that you're completely comfortable with it.

Also, please try to remember that the material in the Math Toolbox is pretty dry and technical, and that it shouldn't be the focus of the proper strategic approach to the SAT. It's just a set of basic ideas that need to be refreshed before we get into the stuff that's more important from a test-taking perspective.

Properties of Integers

An **integer** is any number that can be expressed without a fraction, decimal, percentage sign, or symbol.

Integers can be **positive** or **negative**.

Zero is an integer.

> These numbers are integers: $-99, -6, 0, 8, 675$
>
> These numbers are NOT integers: $\pi, 96.7, \frac{3}{4}$

There are **even** integers and there are **odd** integers.

Only integers can be odd or even—a fraction or symbolic number is neither odd nor even.

Integers that are even can be divided by 2 without having anything left over.

Integers that are odd have a remainder of 1 when they're divided by 2.

> These are even integers: $-6, 4, 8$
>
> These are odd integers: $-99, 25, 671$

Some integers have special properties when it comes to addition and multiplication:

Multiplying any number by 1 leaves the number unchanged.

Dividing any number by 1 leaves the number unchanged.

Multiplying any number by 0 results in the number 0.

Adding 0 to any number leaves the number unchanged.

Subtracting 0 from any number leaves the number unchanged.

It's impossible, for purposes of SAT Math, to divide any number by 0.

Word Problems

SAT **word problems** are typically simple descriptions of real-life situations.

An SAT word problem about a real-life situation might look like this: "Joe buys two balloons for three dollars each, and a certain amount of candy. Each piece of the candy costs 25 cents. Joe gives the cashier ten dollars and receives 25 cents in change. How many pieces of candy did he buy?"

To solve SAT word problems, we sometimes have to transform them into math problems. These are the steps we follow to make that transformation:

- Note all the numbers given in the problem, and write them down on scratch paper.
- Identify key phrases and translate them into mathematical symbols for operations and variables. Use these to connect the numbers you wrote down. After the word problem has been translated into numbers and symbols, solve it like any other SAT Math problem.

In the phrase "two balloons for three dollars each," the *each* part means we have to *multiply* the two balloons by the three dollars in order to find out how much total money was spent on the two balloons. $2 \times \$3.00 = \$6.00$. Six dollars were spent on the two balloons if they cost three dollars each. The sentence "Joe gives the cashier ten dollars and receives 25 cents in change" tells us that we need to *subtract* 25 cents from 10 dollars to find out the total cost of the balloons and candy—that's $\$10.00 - \$0.25 = \$9.75$. If the total cost was \$9.75 and the balloons were \$6.00, then the cost of the candy must have been \$9.75 − \$6.00, or \$3.75. If Joe spent \$3.75 on candy, and each piece of candy cost 25 cents, then we can *divide* \$3.75 by 25 cents to get $\frac{\$3.75}{\$0.25}$, or 15. So Joe bought 15 pieces of candy.

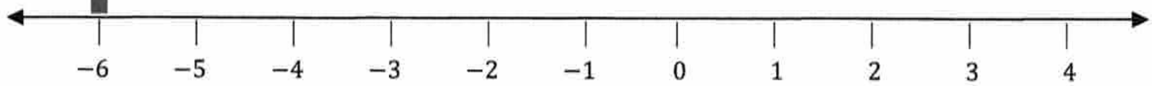
Word problems on the SAT Math section can also occasionally avoid calculations altogether, and be based more heavily on the kinds of careful reading we need to do on the other parts of the SAT. Remember that careful reading is always the single most important skill on every section of the SAT!

Number Lines

A **number line** is a simple diagram that arranges numbers from least to greatest.

The positions on a number line can be labeled with actual numbers or with variables.

This number line shows all the integers from −6 to 4:



On the SAT, number lines are drawn to scale and the tick marks are spaced evenly unless the question notes otherwise.

To determine the distance between two numbers on a number line, just subtract the number to the left from the number to the right.

On the number line above, the distance between 1 and 3 is two units, which is the same thing as saying that $3 - 1 = 2$.

A number's **absolute value** is the distance of that number from zero on the number line.

−4 and 4 both have an absolute value of 4. We signify the absolute value of a number with vertical lines on either side of the number: $|-4| = |4| = 4$

Rounding

SAT Math questions will occasionally ask you to **round** an answer to the nearest whole number, or to the nearest hundredth, or something along those lines. Rounding is a way to make a number a little less accurate, but a little simpler and "cleaner."

To round a number to the nearest whole, for example, we report the value as the whole number closest to the original value.

When we round 64.31 to the nearest whole number, we end up with 64, because 64 is the whole number that is closest to 64.31.

We can also round to the nearest tenth, hundredth, etc.

When we round 7.691 to the nearest tenth, we end up with 7.7, because 7.7 is the "nearest tenth" to 7.691.

When we round 35.376 to the nearest hundredth, we end up with 35.38, because 35.38 is the "nearest hundredth" to 35.376.

By convention, when a number is halfway between two values it might be rounded to, the number is rounded up.

When we round 7.5 to the nearest whole number, we end up with 8, because 7.5 is equally close to 7 and 8, and when the value we want to round is equally close to two possibilities, we round up.

Basic Operations

You'll have to do basic operations (**addition, subtraction, multiplication, division**) with integers, fractions, and decimals.

These are examples of basic operations on integers:

$3 + 4 = 7$ $\quad$ $5 - 2 = 3$

$3 \times 7 = 21$ $\quad$ $8 \div 4 = 2$

These are examples of basic operations on fractions:

$\frac{1}{2} + \frac{3}{2} = 2$ $\quad$ $\frac{4}{3} - \frac{2}{3} = \frac{2}{3}$

$\frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$ $\quad$ $\frac{7}{4} \div \frac{1}{4} = 7$

(We discuss performing basic operations on fractions in more detail in "Fractions and Rational Numbers" on page 170.)

These are examples of basic operations on decimals:

$2.3 + 3.19 = 5.49$ $\quad$ $9.3 - 6.3 = 3$

$1.24 \times 3.5 = 4.34$ $\quad$ $8.7 \div 10 = 0.87$

Squares and Square Roots

To **square** a number, multiply the number by itself.

Five squared is five times five, or 5×5, or 25.

To find the **square root** of a number, find the amount that has to be multiplied by itself in order to generate the number.

The square root of 25 is the amount that yields 25 when it's multiplied by itself. As we just saw, 5 squared is equal to 25. So a square root of 25 is 5.

When you square any number, the result is always positive. This is because a positive number times a positive number gives a positive result, and so does a negative number times a negative number.

A **radical expression** uses the $\sqrt{\ }$ symbol to indicate the square root of a given number.

$\sqrt{7}$ is a radical expression that indicates the square root of 7.

Radical expressions on the SAT are always positive.

Fractions and Rational Numbers

A **fraction** is a special type of number that represents parts of a whole.

Fractions are written this way:

$$\frac{\text{[number of parts being described in the situation]}}{\text{[number of parts that the whole is divided into]}}$$

The number above the fraction bar is called a **numerator.**

The number under the fraction bar is called a **denominator.**

Imagine that we're sharing a six-pack of soda cans. I really like soda, so I drink five of the cans. In this situation, I've had five of the six cans that make up the six-pack—I've had $\frac{5}{6}$ of the six-pack.

When the numerator of a fraction is less than the denominator, the value of the fraction is less than 1.

When the numerator of a fraction is greater than the denominator, the value of the fraction is greater than 1.

$\frac{1}{2}$ is equal to one half, which is less than 1. $\frac{6}{3}$ is equal to 2, which is greater than 1.

Any integer can be thought of as having the denominator 1 already underneath it.

7 is the same thing as $\frac{7}{1}$.

A **reciprocal** is what you get if you switch the numerator and the denominator of a fraction.

The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. The reciprocal of 7 is $\frac{1}{7}$. (Remember that all integers can be thought of as having the denominator 1.)

To multiply two fractions, first multiply their numerators and write that amount as the numerator of the new fraction; then, multiply their denominators and write that amount as the denominator of the new fraction.

$$\frac{4}{7} \times \frac{9}{13} = \frac{36}{91}$$

To divide fraction a by fraction b, we actually multiply fraction a by the RECIPROCAL of fraction b.

$$\frac{4}{7} \div \frac{9}{13} = \frac{4}{7} \times \frac{13}{9} = \frac{52}{63}$$

Fraction a is equal to fraction b if you could multiply the numerator in a by a certain number to get the numerator in b, and you could also multiply the denominator in a by the same number to get the denominator of b.

$\frac{3}{5}$ is equal to $\frac{18}{30}$ because $3 \times 6 = 18$ and $5 \times 6 = 30$. Here's another way to write this: $\frac{3}{5} \times \frac{6}{6} = \frac{18}{30}$. Note that $\frac{6}{6}$ is the same thing as 1 (six parts of a whole that's divided into six parts is the same thing as the whole itself). So all we really did here was multiply $\frac{3}{5}$ by 1, since $\frac{6}{6} = 1$, and we know that doing this will give us an amount equal to $\frac{3}{5}$.

We can **reduce a fraction** when the numerator and denominator can both be evenly divided by the same number—we divide the numerator and denominator by that number. The resulting reduced fraction is equal in value to the original fraction.

The fraction $\frac{15}{25}$ can be reduced because both 15 and 25 can be evenly divided by 5. When we divide the numerator and denominator by 5, we get $\frac{3}{5}$. The reduced fraction $\frac{3}{5}$ is equal to the original fraction $\frac{15}{25}$.

For more on fractions, see the discussion of factors and multiples below.

Factors

The **factors** of a number x are the positive integers that can be multiplied by each other to generate that number x.

The number 10 has the factors 5 and 2, because $5 \times 2 = 10$. It also has the factors 10 and 1, because $1 \times 10 = 10$.

Common factors, as the name suggests, are factors that two numbers have in common.

The number 10 has the factors 1, 2, 5, and 10, as we just saw. The number 28 has the factors 1, 2, 4, 7, 14, and 28. So the common factors of 10 and 28 are 1 and 2, because both 1 and 2 can be multiplied by positive integers to get both 10 and 28.

Prime numbers

A **prime number** is a number that has exactly two factors: 1 and itself.

17 is a prime number because there are no positive integers besides 1 and 17 that can be multiplied by other integers to generate 17. (Try to come up with some—you won't be able to.)

24 is NOT a prime number because there are positive integers besides 1 and 24 that can be multiplied by other integers to generate 24. For example, 2, 3, 4, 6, 8, and 12 can all be multiplied by other integers to generate 24.

All prime numbers are positive.
The only even prime number is 2.
1 is NOT a prime number because it has only one factor (itself), while prime numbers must have exactly two factors.

Multiples

The **multiples** of a number x are the numbers you get when you multiply x by 1, 2, 3, 4, 5, and so on.
The multiples of 4 are 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, and so on.

Order of Operations (PEMDAS)

When an expression involves multiple types of operations, the rules of math require us to perform them in a certain order, called the **order of operations**. Many students learn the proper order by memorizing the acronym **PEMDAS**, which stands for Parentheses, Exponents, Multiplication, Division, Addition, Subtraction.

This is the order in which we must perform the operations in an expression. Any PEMDAS operations that don't appear in an expression are omitted when we evaluate that expression.

Converting among Fractions, Decimals, and Percentages

Because fractions, decimals, and percentages are all ways to express a portion of a whole unit, it can be helpful to know how to express the same value as a fraction, decimal, or percentage.

To make a fraction into a decimal, divide the numerator by the denominator (feel free to use your calculator if you're working on the section that allows you to do that). Remember that your decimal expression will be less than 1 if the numerator is smaller than the denominator, and greater than 1 if the numerator is larger than the denominator.

> For example, $\frac{3}{16} = 3 \div 16 = 0.1875$.

To make a fraction into a percentage, divide the numerator by the denominator, then multiply by 100 and add a percent symbol.

> For example, $\frac{3}{5} = 3 \div 5 = 0.6 = 60\%$.

To make a percentage into a fraction, just give the original percentage a denominator of 100, and then simplify if necessary.

> For example, 47% is the same as $\frac{47}{100}$.

To make a percent value into a decimal expression, just divide the original percentage by 100.

> For example, 3% is the same as 0.03, because $3 \div 100 = 0.03$.

To make a decimal into a percentage, we multiply the original decimal expression by 100 and then add the percent sign.

> For example, 0.895 is the same as 89.5%.

Making a decimal value into a fraction can be a little more complicated, and it doesn't come up very often on the SAT, although the ability to think of a decimal in terms of a roughly equivalent fraction can be helpful sometimes when we're approximating (for example, you probably wouldn't need to know how to convert 0.78 into a fraction, but it may be useful to realize that 0.78 is approximately $\frac{3}{4}$ or $\frac{4}{5}$).

Toward that end, it helps to be able to recognize some decimal expressions and their equivalent fractions:

- $0.1 = \frac{1}{10}$
- $0.\overline{1} = \frac{1}{9}$
- $0.2 = \frac{1}{5}$
- $0.25 = \frac{1}{4}$
- $0.\overline{3} = \frac{1}{3}$
- $0.5 = \frac{1}{2}$
- $0.\overline{6} = \frac{2}{3}$
- $0.75 = \frac{3}{4}$

Remember that you can use your calculator or long division to confirm the fraction equivalent of a decimal expression if you need to for some reason.

> For example, if you see the value 0.125, and you think it's equal to $\frac{1}{8}$, then you can just divide 1 by 8 on a calculator and make sure you're right.

Distance = Rate × Time ($d = rt$)

(This discussion of the distance formula, $d = rt$, includes some relatively basic ideas. But the concepts behind the distance formula can appear on the SAT in different forms, so read this carefully even if you feel comfortable with the distance formula.)

A **rate** is a ratio that tells us how often one event happens in relation to another event happening. Rates are usually (but not always) expressed in terms of time. For example, if someone walks at a rate of 3 miles per hour, then that person walks 3 miles every time 1 hour passes.

To find the distance traveled by an object, we can multiply the rate at which the object travels by the time that the object spends traveling at that rate. As we just saw, this relationship is expressed by the distance formula, $d = rt$.

> What is the distance covered by a car traveling at 60 miles per hour for 2 hours?
>
> $d = rt$ (distance formula)
>
> $d = (60)(2)$ (plug in $r = 60$ and $t = 2$)
>
> $d = 120$ (simplify)
>
> So a car traveling at 60 miles per hour for 2 hours will cover 120 miles.

We can use the same formula to find out how long an object takes to cover a certain distance in a given time:

> How long does it take a car traveling at 40 miles per hour to go 10 miles?
>
> $d = rt$ (distance formula)

$10 = (40)t$ (plug in $d = 10$ and $r = 40$)

$\frac{10}{40} = t$ (divide both sides by 40)

$\frac{1}{4} = t$ (reduce)

So a car traveling at 40 miles per hour will cover 10 miles in $\frac{1}{4}$ hours.

We can also use this formula to find out the speed of an object that covers a certain distance in a given time.

How fast does a car travel if it covers 200 miles in 4 hours?

$d = rt$ (distance formula)

$200 = r(4)$ (plug in $d = 200$ and $t = 4$)

$50 = r$ (divide both sides by 4)

So a car that goes 200 miles in 4 hours travels at an average speed of 50 miles per hour.

The same relationship can be used to discuss the rates at which other processes happen, not just the rate of movement of an object. We just use the variables in the following way:

- We make d equal to the total amount of whatever ends up getting done
- We make r equal to the rate at which that activity gets done
- We make t equal to the amount of time it takes to get that activity done.

Brandon can mow 14 lawns in a week. How long will it take him to mow 168 lawns?

$d = rt$ (distance formula)

$168 = (14)t$ (plug in $d = 168$ and $r = 14$)

$\frac{168}{14} = t$ (divide both sides by 14)

$12 = t$ (reduce)

So it will take Brandon 12 weeks to mow 168 lawns.

(Notice in the example above that the d value was equal to the total number of lawns that got mowed, and the r or "rate" value was the number of lawns Brandon could mow in a week.)

Maritza can write 3 chapters of her book in one month. How many chapters can she write in eight months?

$d = rt$ (distance formula)

$d = (3)(8)$ (plug in $r = 3$ and $t = 8$)

$d = 24$ (simplify)

So Maritza can write 24 chapters in eight months. (Notice in this example above that the d value was equal to the total number of chapters Maritza wrote, and the r or "rate" value was the number of chapters Maritza could write in a month.)

Discounts and taxes

Some SAT word problem involves calculating a **discount** and/or **tax** on some item in a store. When we encounter these problems, we must make sure that we're applying the discount and/or tax to the correct number.

Sven sees a rocking horse for sale for \$150. The store is offering a 25% discount on everything in the store. There is a 10% sales tax that applies to the discounted price of any item in the store. How much does Sven pay for the rocking horse?

The original price of the rocking horse is \$150. To find the price of the rocking horse after the store-wide 25% discount, we multiply \$150 by 0.75, because $100\% - 25\% = 75\% = 0.75$. We find that $\$150 \times 0.75 = \$112.50$. Now we have to calculate the sales tax, which is based on the *discounted price* of the item. To find the price of the rocking horse after the 10% sales tax, we multiply \$112.50 by 1.1, because

$100\% + 10\% = 110\% = 1.1$. We find that $\$112.50 \times 1.1 = \$123.75$, so the final price that Sven pays is $\$123.75$.

Notice that we can't simply apply a discount of 15% based on the idea of subtracting 25% and then adding 10%! The 25% is calculated from the *original* price, while the 10% is based on the *discounted* price, so the percentages can't be combined like this.

Unit conversion

On the SAT, we occasionally encounter questions that require us to convert data from one unit of measure to another unit of measure.

Joe has 500 grams of sugar. If there are 1000 grams in a kilogram and approximately 0.45 kilograms in a pound, about how many pounds of sugar does Joe have?

First, we convert from grams to kilograms. We are told that 1000 grams corresponds to 1 kilogram, and we want to figure out how many kilograms 500 grams corresponds to. So we set up an equation using these ratios:

$\frac{1000\text{ g}}{1\text{ k}} = \frac{500\text{ g}}{x\ k}$ (set ratios equal to one another)

$1000x = 500$ (cross-multiply)

$x = \frac{1}{2}$ (divide both sides by 1000)

So 500 grams is equal to $\frac{1}{2}$ kilogram, or 0.5 kilograms. Now we need to convert kilograms to pounds. We are told that 0.45 kilograms corresponds to approximately 1 pound, and we want to figure out how many pounds 0.5 kilograms corresponds to. So we set up an equation using these ratios:

$\frac{0.45\text{ k}}{1\text{ lb}} = \frac{0.5\text{ k}}{x\ lb}$ (set ratios equal to one another)

$0.45x = 0.5$ (cross-multiply)

$x \approx 1.11$ (divide both sides by 0.45)

So 500 grams is equal to approximately 1.11 pounds.

Note that we can be expected to know the relationships between common, everyday units like minutes, hours, and days—there are 60 minutes in an hour and 24 hours in a day, for example—but any relevant information about more obscure units will be provided by the College Board.

Imaginary Numbers

Imaginary numbers involve the imaginary quantity i or the square root of a negative number. The quantity i represents the square root of -1. The following numbers are all imaginary:

$\sqrt{-7}, 14i, i$

To manipulate expressions and calculations involving i, just treat i like any regular variable, except that i^2 always becomes -1. See the following examples:

$3i + i = 4i$

$2i \times 5i = 10i^2 = 10(-1) = -10$

Real Numbers

On the SAT, a **real number** is any number that doesn't involve i or the square root of a negative number. The following numbers are all real:

$\sqrt{5}, 19, \pi, -4$

Complex Numbers

A **complex number** is an expression with both a real component and an imaginary component, like the following:

$4 + i$

$6 - 3i$

As with imaginary numbers, we can perform operations with complex numbers by treating i like any other variable, and plugging in -1 for i^2, as in the following example:

$3i(2 + i)$	(initial example expression)
$6i + 3i^2$	(distribute $3i$)
$6i + 3(-1)$	(substitute $i^2 = 1$)
$6i - 3$	(simplify)

Complex conjugates

For our purposes, the **complex conjugate** of a complex number is another complex number whose real component is identical but whose imaginary component has the opposite sign but is otherwise identical.

For example, if we're given the imaginary number $5 + 2i$, then its complex conjugate is $5 - 2i$.

Complex conjugates don't come up very often on the SAT Math section. When they do, it's usually in the context of allowing us to remove the imaginary component of a complex denominator. To do this, we take the fraction with the complex denominator and multiply it by a fraction whose numerator and denominator are both equal to the complex conjugate of the denominator of the first fraction, like this:

$\frac{9}{5+2i}$	(initial example expression with complex denominator)
$\frac{9}{5+2i} \times \frac{5-2i}{5-2i}$	(multiply by $\frac{5-2i}{5-2i}$, which is equal to 1)
$\frac{45-18i}{25-4(i^2)}$	(distribute 9 in the numerator; multiply $5 + 2i$ and $5 - 2i$ in the denominator)
$\frac{45-18i}{25-4(-1)}$	(substitute $i^2 = -1$)
$\frac{45-18i}{25+4}$	(simplify)
$\frac{45-18i}{29}$	(simplify)

Set Notation

In the context of SAT Math, a **set** is a group of specific things—usually a group of numbers. In set notation, there are curly braces on either end of the set, and commas between the elements of the set.

The set of positive integers less than 4 is as follows: $\{1, 2, 3\}$

The set of numbers that satisfy the equation $x^2 - 5x - 14 = 0$ is as follows: {-2, 7}

Algebra

For our purposes, **algebra** is the process of using variables like x to stand for unknown numbers in mathematical expressions, and then manipulating those expressions to find the values of one or more of those unknown numbers.

Using equations

On the SAT, an **equation** is a statement that involves an algebraic expression and an equals sign.

$5x = 20$ is an equation, because it involves the algebraic expression $5x$ and an equals sign.

Solving an equation means figuring out the value of the variable in the equation. We solve equations just like you learned in algebra class—by performing the same operations on both sides of the equation until we're left with a value for the variable.

Here's an example of solving for x in an algebraic equation:

$5x = 20$	(example equation)
$\frac{5x}{5} = \frac{20}{5}$	(divide both sides by 5)

$x = 4$ (simplify)

On the SAT, it can often be useful to **cross-multiply**. When we encounter an equation on the SAT with one fraction on either side of the equals sign, we can multiply the denominator of each side by the numerator of the other side and end up with two expressions that must be equal. Let's look at an example without variables first:

Imagine the equation $\frac{9}{15} = \frac{3}{5}$. We know this equation is true because we can reduce $\frac{9}{15}$ to $\frac{3}{5}$ by dividing the numerator and denominator of $\frac{9}{15}$ by 3. Notice that when we cross-multiply—that is, multiply the numerator on the left side of the equation by the denominator on the right side of the equation, and vice-versa—we end up with two expressions that are equal to each other: $9 \times 5 = 3 \times 15$, or $45 = 45$.

This can be a handy shortcut when dealing with an equation involving variables, if the equation has one fraction on each side:

$\frac{12}{x} = \frac{20}{5}$ (given equation)

$12(5) = 20(x)$ (cross-multiply)

$60 = 20x$ (simplify)

$3 = x$ (divide both sides by 20)

Solving an equation for one variable "in terms of" another variable means isolating the first variable on one side of the equation.

What if we have the equation $4n - 7 = 2a$, and we want to solve for n in terms of a?

$4n - 7 = 2a$ (original equation)

$4n = 2a + 7$ (add 7 to both sides to isolate the term involving n)

$n = \frac{2a + 7}{4}$ (divide by 4 to isolate n completely)

Sometimes you'll have a system of equations. A **system of equations** contains two or more equations with the same variables.

A solution to a system of equations is a set of values that creates a valid statement when plugged into each equation in the system. The easiest way to solve a system of equations algebraically is usually to solve one equation in terms of one variable, like we just did before. Then we substitute that value in the second equation and solve.

Imagine that we need to solve the following system of equations, finding an (x, y) pair that satisfies both equations:

$x + y = 5$

$2x - y = 7$

First, we'll isolate the y in the first equation, giving us that equation in terms of y. We can do that by subtracting x from both sides, which gives us $y = 5 - x$. Now that we know y is the same thing as $5 - x$, we just plug in $5 - x$ wherever y appears in the second equation:

$2x - (5 - x) = 7$ (plug $y = 5 - x$ into second equation)

$2x - 5 + x = 7$ (distribute negative sign)

$3x - 5 = 7$ (simplify)

$3x = 12$ (add 5 to both sides)

$x = 4$ (divide both sides by 3)

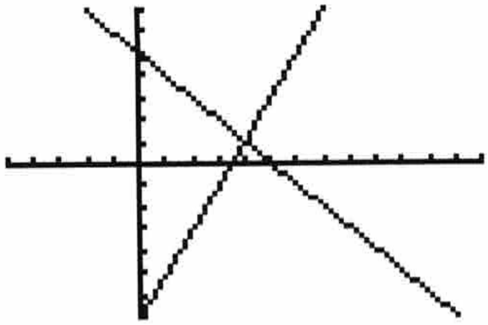
Now that we know x is 4, we just plug that back into either original equation, and we'll be able to solve for y:

$4 + y = 5$ (plug $x = 4$ into first equation)

$y = 1$ (subtract 4 from both sides)

So the solution to the system of equations above is $x = 4$ and $y = 1$ or $(4, 1)$.

When we graph a system of equations, each solution to the system is a point of intersection of the two graphs. This is the graph of the system of equations from the example above:



$x + y = 5$ and $2x - y = 7$

The point of intersection in the graph is $(4, 1)$, the solution of the system of equations.

Inequalities

On the SAT, **inequalities** are statements that show a particular amount is greater than or less than a second amount. They use these symbols:

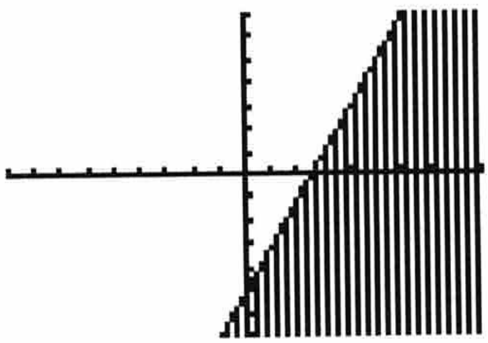
- The symbol $<$ means "less than."
- The symbol $>$ means "greater than."
- The symbol $\leq$ means "less than or equal to."
- The symbol $\geq$ means "greater than or equal to."
- The phrase "from 0 to 10, inclusive," means that 0 and 10 are included in the range.
- The phrase "from 0 to 10, exclusive," means that 0 and 10 are *not* included in the range.

You solve an inequality the same way you solve an equation, with one difference: when you multiply by -1 to solve for a variable, you have to switch the direction of the inequality symbol:

$-\frac{x}{4} \leq 10$	(example inequality)
$-x \leq 10(4)$	(multiply both sides by 4)
$-x \leq 40$	(simplify)
$x \geq -40$	(multiply by - 1 and switch direction of inequality symbol)

When we graph an inequality, we end up with a shaded region on the side of the line that contains all the points that are solutions to the inequality.

This is the graph of $y \leq 2x - 5$. The shaded region represents all the (x, y) coordinates that are solutions of the inequality.



We can check whether a particular point is a solution of an inequality by plugging that point into that inequality. If the result is a valid statement, then the point is a solution of the inequality.

Is the point $(10, 2)$ a solution for $y \leq 2x - 5$?

$y \leq 2x - 5$ (given inequality)

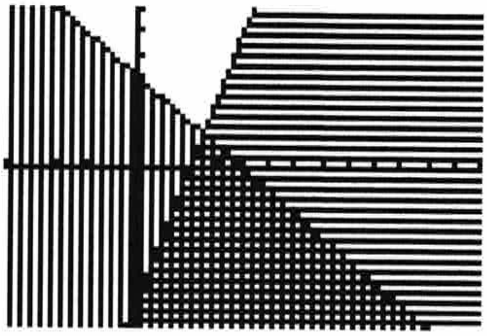
$2 \leq 2(10) - 5$ (plug in $(10, 2)$)

$2 \leq 20 - 5$ (simplify)

$2 \leq 15$ (simplify)

Plugging $(10, 2)$ into the inequality gives us a valid statement. So $(10, 2)$ is a solution of $y \leq 2x - 5$.

We've already discussed systems of equations in this toolbox. We can also have a **system of inequalities**. When we graph a system of inequalities, the solution region will be the set of points in the coordinate plane that satisfy both inequalities.



The above graph represents the following system of inequalities:

$y \leq 4 - x$

$y \leq 3x - 6$

The bottom-middle area of the graph where the two shaded regions overlap is the solution region of this system of inequalities. Each of the points in that double-shaded region satisfies both inequalities.

We can check whether a particular point is a solution to a system of inequalities by plugging that point into each of the inequalities in the system. If the point satisfies all the inequalities, then the point is a solution to the system of inequalities. If the point fails to satisfy all the inequalities, then the point isn't a solution to the system of inequalities.

Is the point $(2, 1)$ a solution for the following system of inequalities?

$y \leq 4 - x$

$y \leq 3x - 6$

First, let's plug $(2, 1)$ into the first inequality.

$y \leq 4 - x$ (first inequality)

$1 \leq 4 - 2$ (plug in $(2, 1)$)

$1 \leq 2$ (simplify)

When we plug $(2, 1)$ into the first inequality, we get a valid statement. So $(2, 1)$ is a solution of the first inequality. Now let's plug $(2, 1)$ into the second inequality.

$y \leq 3x - 6$ (first inequality)

$1 \leq 3(2) - 6$ (plug in $(2, 1)$)

$1 \leq 6 - 6$ (simplify)

$1 \leq 0$ (simplify)

When we plug $(2, 1)$ into the second inequality, we get an invalid statement. So $(2, 1)$ isn't a solution of the second inequality, and it isn't a solution of the system of inequalities.

Exponents

An **exponent** of a number is what we get when we multiply the number by itself a certain number of times.

> a^3 is an example of an exponential expression. It's equal to $a \times a \times a$. The 3 in this example is the "exponent," and the a is called the "base."

Exponents can be positive or negative.

When an exponent is positive, we multiply the base by itself as many times as the exponent indicates, just like we did in the above example.

When an exponent is negative, we treat it just like a positive exponent EXCEPT that we take the reciprocal of the final amount, as demonstrated in the following example:

$$a^5 = a \times a \times a \times a \times a$$

$$a^{-5} = \frac{1}{a^5}$$

We can multiply exponent expressions by each other when the bases are identical. To do that, we just add the exponents:

$$(a^2)(a^3) = (a \times a)(a \times a \times a) = a \times a \times a \times a \times a = a^5$$

$$(a^7)(a^{-4}) = a^3$$

We can also divide exponent expressions when they have the same base. For that we just subtract the exponents:

$$\frac{a^8}{a^2} = a^6$$

We can raise exponential expressions to other exponents by multiplying the first exponent by the second one:

$$(a^4)^5 = a^{20}$$

When we raise a number to a fractional exponent, the denominator of the fraction denotes the root of the base in the expression:

$$a^{\frac{1}{2}} = \sqrt{a}$$

$$a^{\frac{1}{3}} = \sqrt[3]{a}$$

When fractional exponents include numerators other than 1, the numerator determines the exponent to which the base is raised, and the denominator indicates the root of the expression to be taken.

$$a^{\frac{5}{3}} = \sqrt[3]{(a^5)} = \left(\sqrt[3]{a}\right)^5$$

> a raised to the $\frac{5}{3}$ exponent is equal to the cube root of a raised to the fifth power. Notice that it doesn't matter whether the cube root of a is found first and the result is raised to the fifth power, or a is raised to the fifth power and the cube root of the result is found; the value of the expression is the same either way.

Note that raising any number to an exponent of zero gives you the number 1.

$$a^0 = 1$$

Growth and decay

Questions on the SAT will occasionally involve using exponents to calculate **growth** and **decay**. Often, these problems involve money that grows by earning interest, but they can also involve substances other than money, and the amount of that substance can decay (shrink) rather than grow.

If answering a growth or decay question requires the use of a formula, then that formula will be provided, and it will look something like this:

$$\textit{amount of substance} = \textit{initial amount of substance}(\textit{rate of growth or decay})^{\textit{amount of time}}$$

When you plug values into each of the three components on the right hand side of the formula, you will get the amount of the substance after a given period of time at the provided rate of growth or decay.

If the question were related to money, the result might be the amount of money that results from depositing an initial amount of money for a certain number of years at a given interest rate. If the question were about radioactive decay, the result might be the amount of an element that remains after a certain number of years if the amount of the substance decreases by a given percentage each year, and so on.

Either way, the test will provide a formula in the question if a formula is necessary; otherwise, you might be asked a question that requires a basic understanding of how a growth or decay formula works, as we've discussed in this section and will see in the relevant walkthroughs later in this book.

Polynomials

On the SAT, a **polynomial** is an expression that includes multiple terms, at least one of which contains a variable.

> The following are examples of polynomials:
>
> $5x + 3$
>
> $x^2 - 2x + 9$
>
> $4ab^2 + 5a$

Remember that a **variable** is a letter, like x or a, that's used to represent a value. In different expressions, a variable can have different values. For example, in one expression, x can be equal to 100, while in another expression, x can be equal to 12.

A **constant** is a number like 3 or 11 or 716.9234 that has a set, defined value that can't change. The constant 7 is always equal to 7, regardless of the expression where the 7 appears.

A **coefficient** is a constant that appears immediately before a variable in a polynomial to indicate that the variable is being multiplied by the amount of the coefficient. So in the expression $8 + 2y$, we know that 8 is a constant, y is a variable, and 2 is the coefficient of y.

When we write a polynomial, we list the terms in decreasing order of the exponent of the variable. That sounds complicated, so let's look at an example:

> $x^4 + 3x^3 + 5x^2 - 2x + 4 = 0$

We started with the x-term raised to the fourth power, then followed with the one raised to the third power, then the one that was squared, then the x-term without an exponent (which is the same as x^1), then the constant with no x-term (which as the same as an x^0 term, since any number raised to the power of zero is equal to 1).

The **degree** of a variable in a polynomial refers to the highest exponent to which that variable is raised in the expression. So $x^3 + 5x^2 - x - 7$ is a third-degree polynomial, because the highest exponent of an x-term in the polynomial is 3.

Operations on polynomials

Polynomials can be added, subtracted, multiplied, and divided, but sometimes there are special considerations.

When we add or subtract polynomials, we simply combine like terms.

> Let's add the polynomials $5x + 3$ and $x^2 - 2x + 9$:
>
> $5x + 3 + x^2 - 2x + 9$ (original expression)
>
> $x^2 + (5x - 2x) + (3 + 9)$ (group like terms)
>
> $x^2 + 3x + 12$ (simplify)
>
> As you can see, we combined the following kinds of terms:
>
> - x^2 terms (there was only one, in this case)
> - the x terms ($5x$ and $-2x$)
> - the constants (3 and 9)
>
> Notice that you can only add or subtract like terms during this process.
>
> Let's add the polynomials $2x^3 + 4x^2$ and $x + 7$:
>
> $2x^3 + 4x^2 + x + 7$ (given expression)
>
> In this case, nothing can be combined, because there are no like terms. So we just write the sum as shown above.

We can multiply a polynomial by another quantity by multiplying each term in the polynomial by that quantity.

> $3y(5x + 6)$ (given expression)

Suppose we're asked to factor the expression $6x^2 - 7x - 3$ into two binomials. We'll need to reverse-FOIL the expression. We can see the product of the first terms of each polynomial will have to be $6x^2$. So we'll just pick two x terms to try out, like $3x$ and $2x$:

$$(3x+?)(2x+?)$$

We also know the last terms will have to multiply together to equal -3, so they must be either -1 and 3 or 1 and -3. We'll just try one pair (reverse-FOILing often involves some trial and error).

$$(3x+1)(2x-3)$$

Then we multiply out our binomials to see if we've reverse-FOILed correctly:

$(3x+1)(2x-3)$	(our guess for the factorization)
$6x^2 - 9x + 2x - 3$	(FOIL the two binomials)
$6x^2 - 7x - 3$	(simplify)

In this case, we got it right on the first try; if you don't, of course, you can look at what didn't work and try other pairs of factors. Problems like this on the SAT aren't too common, and the factors usually aren't too hard to figure out. This gets a lot easier with a little bit of practice.

A **difference of squares** is a special case in factoring binomials. You can recognize a difference of squares because both terms in the binomial will be squares, and the second term will be subtracted from the first (this is why it's called a "difference" of squares; this special factoring shortcut doesn't work when the squares are added together). When we see this situation, the two factors are always the following:

- the square root of the first term *plus* the square root of the second term
- the square root of the first term *minus* the square root of the second term

$$4x^2 - 25 = (2x+5)(2x-5)$$

Remember 1 is square, and x^2 equals $1x^2$. So an expression like $x^2 - 9$ is a difference of squares, with factors of $(x+3)$ and $(x-3)$.

"Zeros" of a polynomial function

The SAT sometimes asks us about the **zeros** of a function that involves a polynomial. The term "zero of a polynomial" refers to a value of a variable (usually x) that makes the polynomial expression equal to zero.

We can see that $x = 1$ is a zero of the polynomial $x^2 + 2x - 3$:

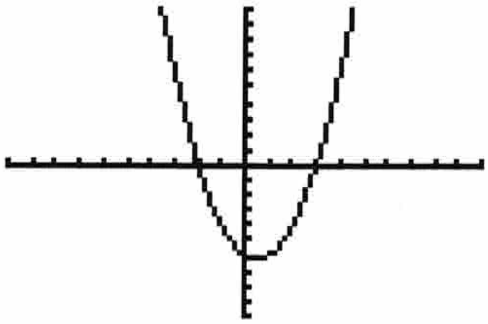
$x^2 + 2x - 3$	(given polynomial)
$(1)^2 + 2(1) - 3$	(plug in $x = 1$)
$1 + 2 - 3$	(simplify)
0	(simplify)

Another way to find the zeros of a polynomial involves factoring it. Once we know the factors of a polynomial, we can set them each equal to zero to find the zeros of the polynomial (because if any of the factors equals zero, then the whole expression must equal zero).

$x^2 + 2x - 3 = 0$	(original polynomial, set equal to zero)
$(x+3)(x-1) = 0$	(factor expression by reverse-FOIL)
$x + 3 = 0$ or $x - 1 = 0$	(identify the 2 possibilities to make the original expression equal 0)
$x = -3$ or $x = 1$	(solve to find the two zeros of the original polynomial)

We've found that $x = -3$ and $x = 1$ are both zeros of the polynomial, because the factors are $(x+3)$ and $(x-1)$; if either of those factors is equal to zero, then the whole expression must equal zero. Note that we can also work in the other direction: if we know that -3 is a zero of a polynomial, then we know that one factor of that polynomial is $(x-(-3))$, or $(x+3)$. Similarly, if we know that 1 is a zero of a polynomial, then we know that $(x-1)$ is a factor of that polynomial.

Another way to find the zeros of a polynomial function is to use a graphing calculator. When we graph a function, the zeros of that function are the points where the graph of the function touches or crosses the x-axis:



The above is the graph of $f(x) = x^2 - x - 6$. The graph crosses the x-axis at $x = -2$ and $x = 3$, so we know the zeros of the function are $x = -2$ and $x = 3$, and the factors are $x + 2$ and $x - 3$.

Completing the square

Sometimes it can be useful to rewrite a polynomial expression as an equivalent polynomial that includes a squared binomial, which is a technique called "completing the square." This is especially true in the context of equations for circles, parabolas, lines, and other conic sections, because completing the square allows us to produce standard versions of those equations that can give us useful information about key features of the conic section.

We complete the square by adding the same amount to both sides of a given equation so that one side becomes a polynomial expression that can be expressed as the square of a binomial; we then reverse-FOIL that expression so that we can write it as the square of a binomial, and then finish the process by re-isolating the other variable.

Let's complete the square so that we can create an expression of the equation $y = x^2 + 6x + 15$ that involves a squared binomial.

$y = x^2 + 6x + 15$	(original polynomial, not expressed as the square of a binomial)
$y - 15 = x^2 + 6x$	(subtract the constant on the right side from both sides)
$y - 15 + 9 = x^2 + 6x + 9$	(find half the coefficient of x, square the result, and add that number to both sides of the equation)
$y - 6 = (x + 3)^2$	(simplify on the left side of the equation, and reverse-FOIL the right side of the equation to generate the squared binomial)
$y = (x + 3)^2 + 6$	(add or subtract any constants on the left to both sides, as necessary, to isolate y again)

So when we complete the square with the equation $y = x^2 + 6x + 15$, the result is $y = (x + 3)^2 + 6$.

Polynomial long division

This is similar to the long division you may have learned in elementary school or middle school, but it involves polynomials instead of constants.

(NOTE: This is a relatively obscure math concept that only appears rarely on the SAT! If you're struggling with it, you may find that it's not worth the trouble to keep working on it too much: there's a good chance you won't see it at all on test day, and, even if you do, it's still possible to get a perfect or near-perfect SAT Math score if you miss one question on the Math section. It may also be possible to answer a question that seems to involve polynomial long division without actually doing the polynomial long division, as we'll see in the walkthrough for question 15 on section 3 of SAT Practice Test #2 on page 269 later in this book.)

The steps of polynomial long division are essentially the same as the steps of regular long division. The best way to explain them is probably with an example:

Let's divide $6x - 15$ by $x + 2$. In this case, $6x - 15$ is the numerator, and $x + 2$ is the denominator. First, we set up our work like this:

$$x+2\overline{)6x-15}$$

Next, we figure out how many times the *first* term in the denominator can go into the *first* term in the numerator. In this case, that means figuring out how many times x goes into $6x$. We write in that number above the numerator, and we write in the result of multiplying the denominator by that number below the numerator. In this case, x goes into $6x$ exactly 6 times, so we write in 6 above the numerator. $x + 2$ multiplied by 6 equals $6x + 12$, so we write in $6x + 12$ below the numerator:

$$\begin{array}{r} 6 \\ x+2\overline{)6x-15} \\ 6x+12 \end{array}$$

Now we subtract the number below the numerator from the numerator, and write the result below.

$$\begin{array}{r} 6 \\ x+2\overline{)6x-15} \\ \underline{-(6x+12)} \\ 0-27 \end{array}$$

(Notice that $6x$ and 12 are both positive numbers, but we <u>subtract</u> those numbers from the numbers directly above them to find that $6x + 12$ subtracted from $6x - 15$ equals -27.)

Next, we see whether the first term in the denominator can go into the first term in the number we just found. If it can, we repeat the process again until the result of our subtraction is 0, or until we find that the first term in the denominator can no longer go into the first term at the bottom of our work.

In this case, the variable x can't go into the constant -27, so -27 becomes part of the remainder. We make a fraction with - 27 as the numerator and the original denominator as the denominator, and we add the result to the number above the numerator.

$$6 + \frac{-27}{x+2} \text{ or } 6 - \frac{27}{x+2}$$

So the result when we divide $6x + 15$ by $x + 2$ is $6 - \frac{27}{x+2}$.

This topic is particularly awkward to explain, so you may want to read it through slowly another couple of times, thinking about each step and matching up the description of the step with the relevant work in the example.

The Remainder Theorem

The Remainder Theorem says that when we divide a polynomial function $p(x)$ by $x - c$, the remainder is equal to $p(c)$. This probably sounds complicated in the abstract, so let's look at an example:

The Remainder Theorem tells us that dividing $p(x) = x^3 - 5$ by $x - 2$ results in a remainder of 3, because $p(2) = 3$.

Note that you probably don't need to know the Remainder Theorem on test day, because it comes up very, very rarely, and even when it does come up, you can generally work around it—and, if you do ever need it, you'll only need to know the very basic application we've covered here.

Solving Quadratic Equations by Factoring

A quadratic equation is an equation that involves three terms:

- one term is a variable expression raised to the power of 2.
- one term is a variable expression not raised to any power.
- one term is a constant.

$x^2 + 3x = -2$ is a quadratic equation because it involves a term with x squared, a term with x, and a constant.

One way to answer SAT questions that involve quadratic equations is by factoring. See the discussion of factoring polynomials above if you need a refresher on that general idea. (Graphing and the quadratic formula are two other ways to answer SAT questions that involve quadratic equations, in case you were wondering.)

To solve a quadratic equation by factoring, we have to make one side of the equation equal to zero, and then factor the other side of the equation (the quadratic part).

$x^2 + 3x = -2$ (given quadratic equation)

$x^2 + 3x + 2 = 0$ (add 2 to both sides to make right side equal to 0)

$(x + 1)(x + 2) = 0$ (reverse-FOIL left side of equation)

Now that we know $(x + 1)(x + 2) = 0$, what else do we know? We know that one of those two factors has to equal zero—either $x + 1 = 0$ or $x + 2 = 0$. How do we know this? Remember that the only way to multiply two numbers and get zero is if one of the numbers is zero. So if we can multiply $x + 1$ by $x + 2$ and get zero, then either $x + 1$ is zero or $x + 2$ is zero.

Once we've factored, we solve for the variable by creating two small sub-equations in which each factor is set equal to zero.

$x + 1 = 0$ or $x + 2 = 0$

$x = -1$ or $x = -2$

So in the equation $x^2 + 3x = -2$, x can equal either -1 or -2.

Quadratic equations can have multiple solutions, as we've just seen.

Quadratic formula

The general quadratic equation is $ax^2 + bx + c = 0$. As we've discussed elsewhere, it's sometimes possible to determine the value of x in such a quadratic equation through different types of factoring—but those approaches aren't always easy to apply.

We can *always* find the solutions to a quadratic equation using the quadratic formula, which is as follows, given the general quadratic format of $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Find the solutions to the equation $3x^2 + 10x - 2 = 0$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (quadratic equation)

$x = \frac{-(10) \pm \sqrt{(10)^2 - 4(3)(-2)}}{2(3)}$ (plug in $a = 3$, $b = 10$, and $c = -2$)

$x = \frac{-10 \pm \sqrt{124}}{6}$ (simplify)

$x = \frac{-10 \pm 2\sqrt{31}}{6}$ (factor $\sqrt{4}$ out of the expression under the radical)

$x = \frac{-5 \pm \sqrt{31}}{3}$ (reduce)

So the two solutions are $x = -\frac{5}{3} + \frac{\sqrt{31}}{3}$ and $x = -\frac{5}{3} - \frac{\sqrt{31}}{3}$.

(Notice in the example above that we find the a, b, and c values by comparing the given equation to the general quadratic equation. In this case, when we compare the given equation $3x^2 + 10x - 2 = 0$ to the general quadratic equation $ax^2 + bx + c = 0$, we can see that $a = 3$, $b = 10$, and $c = -2$.)

Functions

Concepts related to functions appear frequently on the SAT. Make sure you're thoroughly comfortable with the material in this section before test day, and see the question walkthroughs later in this Black Book for examples of how these concepts are tested in real questions.

A **function** is a type of equation that allows us to enter one value (often called x) and generate another value (often called y, or $f(x)$).

The values we enter into the function can be referred to as **inputs** or **x-values** (if the function is written in terms of x).

The values generated by the function can be referred to as **outputs** or **y-values** (if the function is written in terms of y) or $f(x)$ values (if the function is called f and written in terms of x-values).

There are many different ways we can write a function, but one of the most common ways we'll encounter on the SAT is probably by using $f(x)$:

$f(x) = x + 3$

Even though the most common way that functions are expressed on the test is in terms of f and x, any other two letters can also be used, as in the following examples:

$g(h) = h + 3$

$E(\theta) = \theta + 3$

$r(N) = N + 3$

All of these functions express the same relationship; they just use different letters as variables.

In the example function above, the function $f(x)$ is equal to $x + 3$. In other words, for any x-value that we enter into the function above, we find $f(x)$ by adding 3 to the x-value. For example, entering $x = 8$ will produce an $f(x)$ value of 11. We could write this in the following way: $f(8) = 11$.

The expression in parentheses immediately after the f is the thing we'll be plugging into the function:

- If the expression in parentheses after the f is a variable, such as x, then we're using it to represent the idea of *any* expression being plugged into the function.
- If the expression in parentheses after the f is a specific value, such as 8, then the notation describes the idea of that specific value being plugged into the function.

$f(x) = x + 3$ (equation defining function f)

$f(8) = 8 + 3$ (plug in $x = 8$)

$f(8) = 11$ (add 3 and 8 to find $f(8)$)

The **domain** of a function is the set of numbers on a number line where the function can be evaluated.

In the function $f(x) = x^3 + 4$, the domain is all the numbers on the number line, because we can plug any value from the number line in for x and get a result for $f(x)$.

In the function $f(x) = \sqrt{x}$, the domain is only those numbers that can have a real square root. Negative numbers don't have real square roots, so the domain for the function $f(x) = \sqrt{x}$ is the set of non-negative numbers.

The **range** of a function is the set of numbers that $f(x)$ can come out equal to.

The function $f(x) = x^3 + 4$ has a range of negative infinity to positive infinity—by plugging in the right value for x, we can get any number we want as $f(x)$.

The function $f(x) = \sqrt{x}$ has a range of only non-negative numbers, because there is no way to put in any number as x and get a number for $f(x)$ that's negative. (Remember that a radical expression is never negative on the SAT.)

A function is **undefined** when the function includes a fraction whose denominator is equal to 0.

The function $f(x) = \frac{1}{x-3}$ is undefined at $x = 3$ because $f(3) = \frac{1}{(3)-3}$ which is equal to $\frac{1}{0}$.

Nested functions

To take the concept of a function a step farther, let's look at the idea of using the output from one function as the input for another function. We'll call this a **nested function.**

Let's imagine that we have one function called $f(x)$, and another function called $g(x)$. Function $f(x)$ will be equal to $x^2 + x + 2$, and function $g(x)$ will be equal to $\frac{x}{3} - 1$:

$f(x) = x^2 + x + 2$ (definition of $f(x)$)

$g(x) = \frac{x}{3} - 1$ (definition of $g(x)$)

We can evaluate an expression like $f(g(x))$ by plugging x into $g(x)$ first, and then taking the result and plugging it into $f(x)$. Let's follow that process to find $f(g(9))$.

$g(x) = \frac{x}{3} - 1$ (definition of $g(x)$)

$g(9) = \frac{(9)}{3} - 1$ (plug $x = 9$ into $g(x)$)

$g(9) = 2$ (simplify)

So we see that $g(9) = 2$. Now we plug 2 into $f(x)$.

$f(x) = x^2 + x + 2$ (definition of $f(x)$)

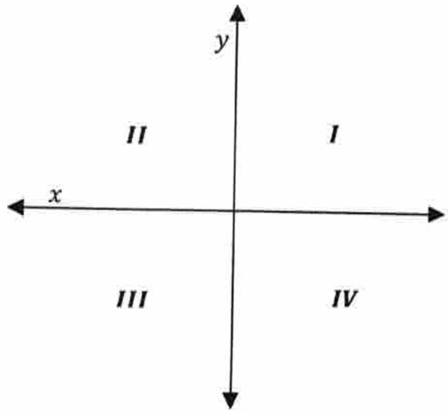
$f(2) = (2)^2 + (2) + 2$ (plug in $x = 2$ into $f(x)$)

$f(2) = 8$ (simplify)

So by finding $g(9)$, then plugging that value into $f(x)$, we find that $f(g(9)) = 8$.

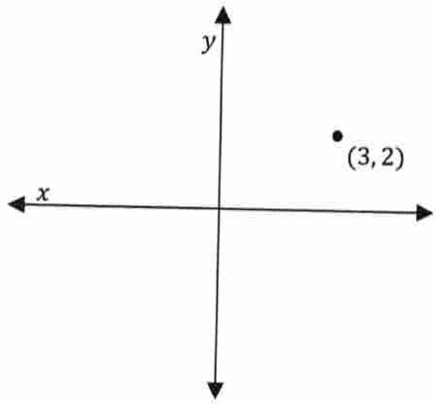
Linear functions

The $\boldsymbol{xy}$**-coordinate plane** has 4 **quadrants** numbered I, II, II, and IV.



The **origin** is the point where the x-axis and y-axis intersect. The coordinate pair that corresponds to the origin is $(0, 0)$.

A **point** can be plotted on the xy-coordinate plane in (x, y) notation if we make the x number the horizontal separation between the point (x, y) and the origin $(0, 0)$, and then we make the y value the vertical separation between (x, y) and $(0, 0)$.



The above graph shows the point $(3, 2)$ on the xy-coordinate plane. (Notice in the graph above that point $(3, 2)$ is 3 units to the right of the origin and 2 units above the origin.)

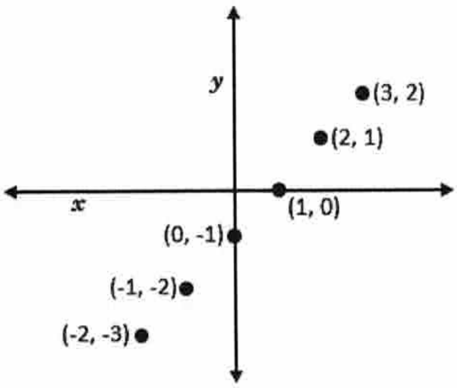
A **linear function** is a function whose (x, y) value pairings form a straight line when they're plotted as points on a graph.

$f(x) = x - 1$ is linear, because all the (x, y) pairings that it generates form a straight line when plotted on a graph.

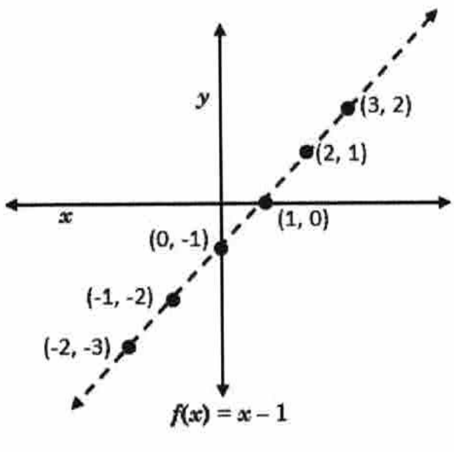
Here are some (x, y) pairings for the function $f(x) = x - 1$:

x	y
-2	-3
-1	-2
0	-1
1	0
2	1
3	2

When we plot the (x, y) pairings from a linear function, we can see they fall in a straight line:



As you can see, we've only plotted six specific points based on the (x, y) coordinates we got for six specific values of x. But we can see that the domain for $f(x) = x - 1$ must be all numbers from negative infinity to infinity, because any x value we plug in will result in a defined y value. So we can draw a line connecting these plotted points, and the line will represent all possible (x, y) pairs that satisfy $f(x) = x - 1$.



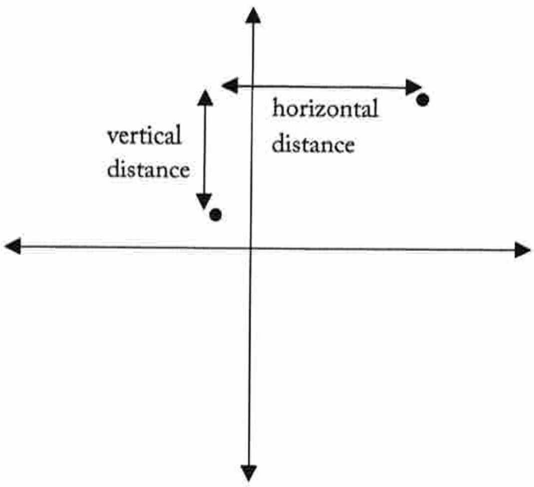
The **slope** of a function is a fraction that expresses a measurement of how far up or down the line travels for a given distance that it travels from left to right:

- The number of units traveled up goes in the numerator of the slope fraction. (A negative number in the numerator indicates that the line travels down as it goes from left to right, and a zero in the numerator—in other words, a slope of zero—indicates that the line is horizontal.)
- The number of units traveled from left to right is the denominator of the slope fraction.

For this reason, the slope fraction is often described as "rise over run"—the number of units traveled up (the "rise"), divided by the number of units traveled from left to right (the "run").

If the slope of a line is $\frac{1}{3}$, then the line travels 1 unit up for every 3 units it travels from left to right, because the numerator, or "rise," is 1, and the denominator, or "run," is 3.

This means we can calculate the slope between two points on a function graph by finding the vertical distance between them, and dividing it by the horizontal distance between them:



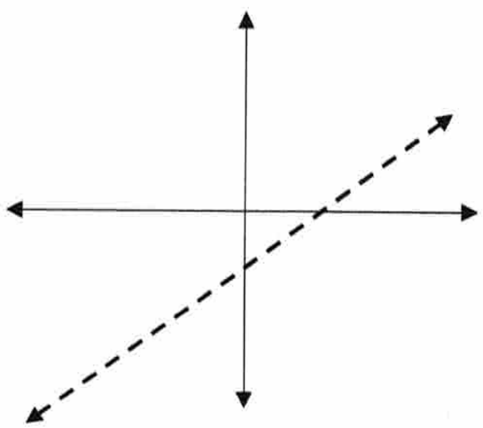
The vertical distance between the two points is the difference between the y-values of the points, and the horizontal distance between them is the difference between the x-values.

> The slope between the points $(5, 3)$ and $(19, 8)$ is equal to $\frac{8-3}{19-5}$, or $\frac{5}{14}$.

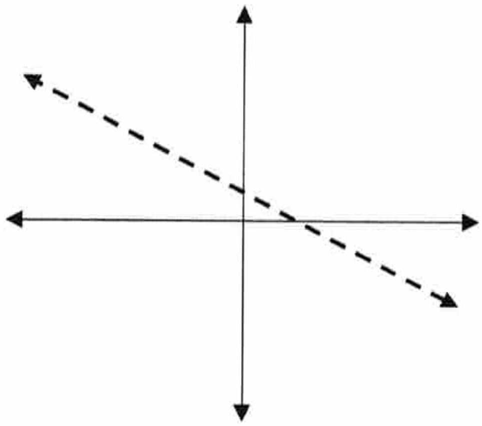
By convention, we take the values from the left-most point and subtract them from the values of the right-most point, but it doesn't actually matter which order you subtract in, as long as you use the same order for coordinates in the numerator and denominator, and you keep track of any minus signs.

> If we switch the order of subtraction for the points from the previous example, we get $\frac{3-8}{5-19}$, or $\frac{-5}{-14}$, which is still equal to $\frac{5}{14}$.

As we discussed earlier, a line with positive slope is slanted upward as we read from left to right, and a line with a negative slope is slanted downward as we read from left to right:



Line with Positive Slope



Line with Negative Slope

The **y-intercept** is the y-value of the function where $x = 0$, and is also the point where the graph of the function crosses the y-axis.

Two linear functions with the same slope and different y-intercepts are **parallel**.

Two lines are **perpendicular** when their slopes are the negative reciprocals of one another.

> 2 and $-\frac{1}{2}$ are examples of perpendicular slopes

The slope of a **horizontal** line is 0, because the "rise" is always zero, and 0 divided by any number is 0.

The slope of a **vertical** line is **undefined**, because the "run" is always zero, and any number divided by 0 is undefined.

You'll never have to draw your own graph of a linear function on the SAT, because the formatting of the test makes that impossible. Instead, you might have to use your understanding of graphs to figure out a value, or to pick one graph out of several others as the correct graph of a particular function. (Your calculator may come in handy for that—see the walkthroughs later in this Black Book for examples of this idea in action.)

Note that an **x-intercept** of a function is a point where the graph of the function crosses the x-axis, which is also a point where the y-value of the function is equal to 0.

Slope-intercept form

We can easily find the slope of a line when the function equation for the line is in something called **slope-intercept form,** which looks like this:

$$y = mx + b$$

The variables in slope-intercept form represent the following aspects of the graph:

- y and x represent the values in an (x, y) coordinate pair of any point on the graph
- m represents the slope, often in fraction form
- b represents the y-intercept

So when a line is in slope-intercept form, the coefficient of x is the slope of the line.

> For the function $y = 4x - 7$, the slope of the line is 4, and the y-intercept is $(0, -7)$.
>
> For the function $y = \frac{x}{5} + 1$, the slope is $\frac{1}{5}$, and the y-intercept is $(0, 1)$. (Remember that $\frac{x}{5}$ is the same as $\frac{1}{5}x$.)

Point-slope form

When we know the slope of a line, m, and a point on that line, (x_1, y_1), we can create an equation for that line using **point-slope form,** which looks like this:

$$y - y_1 = m(x - x_1)$$

> For the line with slope $m = 2$ that contains the point $(1, 6)$, an equation using point-slope form is $y - 6 = 2(x - 1)$.

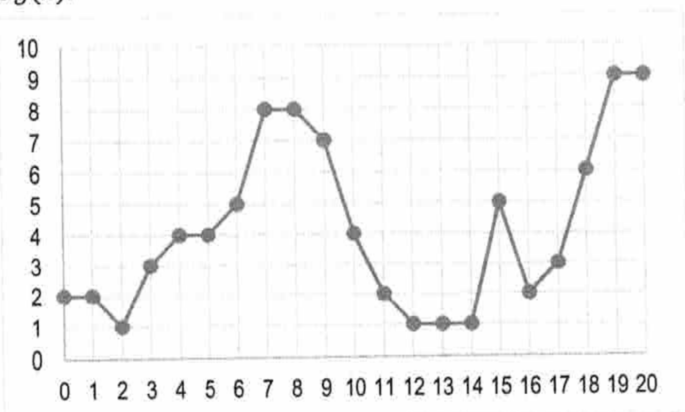
Evaluating a function

When we evaluate a function like $f(x)$, we determine the output when we plug in a certain x value. The two most common ways to evaluate a function are:

- to plug an x value into the function itself, or
- to look at a graph of the function and see which y value corresponds to a certain x value.

We'll look at both methods.

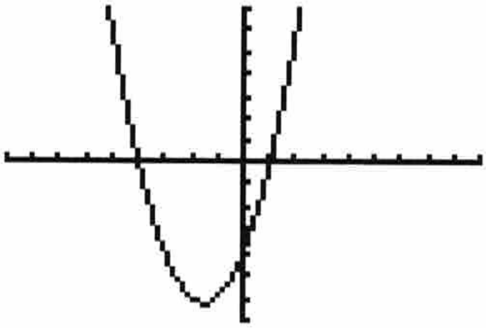
> We can evaluate $f(x) = 2x + 4$ when $x = 5$ by plugging 5 into the given function:
>
> $f(x) = 2x + 4$ (given function)
>
> $f(5) = 2(5) + 4$ (plug in $x = 5$)
>
> $f(5) = 14$ (simplify)
>
> So if we evaluate $f(x) = 2x + 4$ when $x = 5$, we find that $f(5) = 14$.
>
> Below is the graph of $g(x)$:



> We can evaluate $g(5)$ by finding the value of $g(x)$ when $x = 5$. We can see that when $x = 5$, the y-value on the graph is 4. So we know that $g(5) = 4$.
>
> (Notice that when we have a graph of $g(x)$, we can evaluate $g(x)$ without actually knowing what expression $g(x)$ is equal to.)

An **equivalent form** of a function or equation is basically another way to write the same function or equation. Both forms of the function or equation will be satisfied by the exact same set of (x, y) coordinates, which means they produce identical graphs. Any function or equation can have many different equivalent forms.

$y = (x + 4)(x - 1)$ and $y = x^2 + 3x - 4$ are equivalent forms of the same equation, because they're both satisfied by the same sets of (x, y) coordinates, and both produce the same graph, which looks like this:



Quadratic functions

A **quadratic function** is a function where the x variable has an exponent of 2.

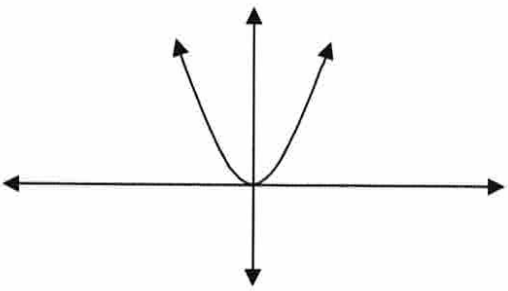
$y = x^2$ is a quadratic function.

Quadratic functions are NEVER linear—in other words, the graph of a quadratic function is always a curve.

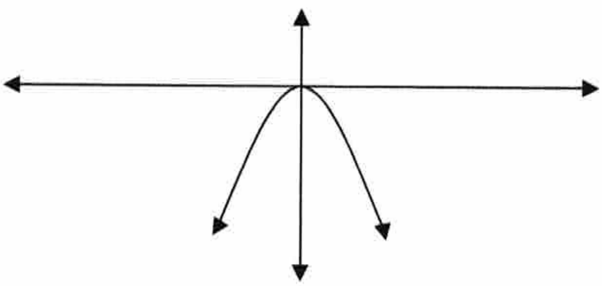
The SAT never requires you to draw a graph by hand. It will only ask you to use given graphs to answer questions, or to identify which answer choice correctly graphs a given function.

Quadratic functions always extend infinitely in some direction (up, down, left, right, or a combination).

The graph of $y = x^2$ extends up infinitely, and looks roughly like this:



The graph of $y = -(x^2)$ extends down infinitely, and looks roughly like this:



Note that in the context of a discussion of functions on the SAT, the "direction" of the graph of a quadratic equation is really just a question of its range. When the range extends to negative infinity, the graph "opens down." When the range extends to positive infinity, the graph "opens up."

When a quadratic function "opens down," its highest point or **maximum value** is the (x, y) pair that has the greatest y value.

When a quadratic function "opens up," its lowest point or **minimum value** is the (x, y) pair that has the lowest y value.

Sometimes you'll be asked to find the "zeros" of a quadratic function. Quadratic functions are a kind of polynomial functions, so this process is the same as finding zeros of a polynomial, as discussed earlier in this Toolbox.

Geometric Notation

The SAT likes to use what it calls "geometric notation" to describe lines, rays, angles, and so on. You've probably seen this notation in your classes, but don't worry if you haven't—it's not hard to learn.

AB describes the distance from A to B.

$\overleftrightarrow{AB}$ describes the line that goes through points A and B (the arrows indicate an infinite extension into space in both directions).

$\overline{AB}$ describes the line segment with endpoints A and B (the lack of arrowheads on the symbol indicates that the given segment doesn't continue on to infinity).

$\overrightarrow{AB}$ describes the ray with endpoint A that goes through B and then continues on infinitely.

$\overrightarrow{BA}$ describes the ray with B for an endpoint that goes through A and continues on infinitely.

$\angle ABC$ describes the angle with point B as a vertex that has point A on one leg and point Con the other.

$\angle ABC = 60°$ indicates that the measure of the angle with point B as a vertex and with point A on one leg and point C on the other is 60 degrees.

$\triangle ABC$ describes the triangle with vertices A, B, and C.

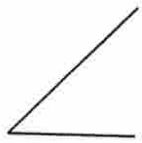
$\square ABCD$ describes the quadrilateral with vertices A, B, C, and D.

$\overline{AB} \perp \overline{BC}$ indicates that the line segments $\overline{AB}$ and $\overline{BC}$ are perpendicular to each other.

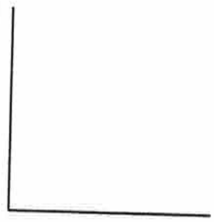
$\overline{AB} \parallel \overline{BC}$ indicates that the line segments $\overline{AB}$ and $\overline{BC}$ are parallel to each other.

Angles in the Plane

Degrees are the units that we use to measure how "wide" or "big" an angle is.
This is a 45-degree angle:



This is a 90-degree angle, also called a "right angle:"



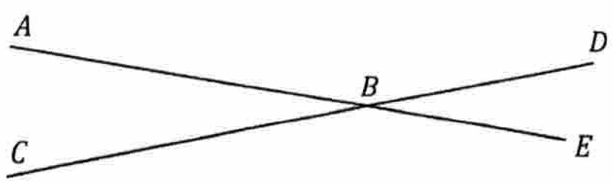
This is a 180-degree angle, which is the same thing as a straight line:

Congruent angles are just angles with the same measures. So if one angle has a measure of 30° and another angle has a measure of 30°, those two angles are congruent.

Sometimes angles have special relationships. Three types of special relationships that appear often on the SAT are vertical angles, supplementary angles, and complementary angles.

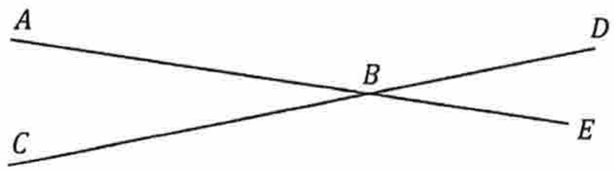
Vertical angles are the pairs of angles that lie across from each other when two lines intersect. In a pair of vertical angles, the two angles have the same degree measurements as each other.

Angles $\angle ABC$ and $\angle DBE$ are a pair of vertical angles, so they have the same degree measurements as each other. Angles $\angle ABD$ and $\angle CBE$ are also a pair of vertical angles, so they have the same measurements as each other as well.



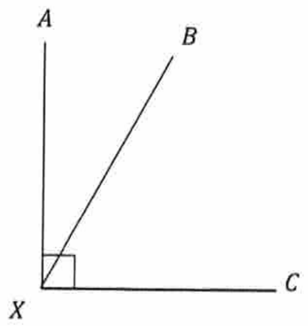
Supplementary angles are pairs of angles whose measurements add up to 180 degrees. When supplementary angles are next to each other, they form a straight line.

$\angle ABC$ and $\angle ABD$ are a pair of supplementary angles, because their measurements together add up to 180 degrees—together, they form the straight line CD.

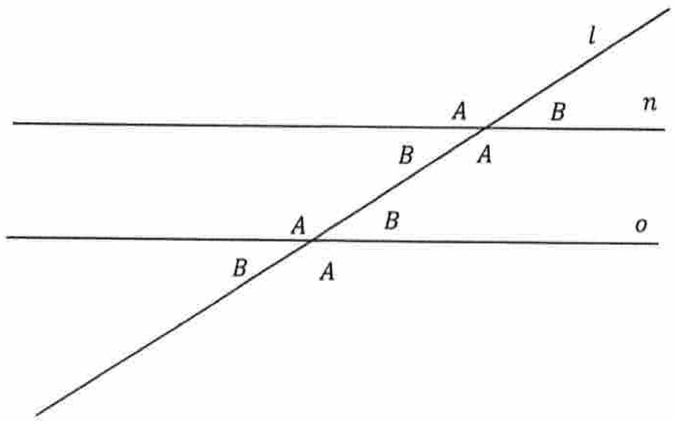


Complementary angles describe angles who measures add up to 90°.

In the figure below, $\angle AXB$ and $\angle BXC$ are complementary, because the sum of the measures of the two angles is 90°.



A **transversal** is the result when a line crosses two parallel lines, as we can see in the diagram below.



In the above diagram,

- n and o are parallel
- all angles labeled A are equal to one another
- all angles labeled B are equal to one another
- $A + B = 180°$

We can see that each angle labeled A lies across from another angle labeled A, and that each pair of angles labeled A is a set of vertical angles. Similarly, each angle labeled B lies across from another angle labeled B, and each pair of angles labeled B is a set of vertical angles.

Also, because lines n and o are parallel, we know that that line l crosses line n at the same angle that line l crosses line o. For this reason, we know that the angles labeled A in the top intersection are equal to the angles labeled A in the bottom intersection, and also that the angles labeled B in the top intersection are equal to the angles labeled B in the bottom intersection.

There are special terms for the relationships among the various angles created by a transversal, but we don't need to know those on the SAT, beyond what we've already discussed in this section.

Triangles

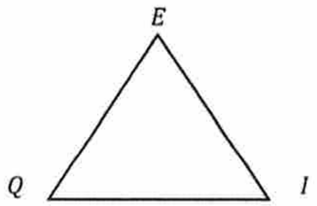
The SAT loves to ask about **triangles**.

The sum of the measures of the angles in any triangle is 180 degrees.

In any triangle, the longest side is always opposite the biggest angle, and the shortest side is always opposite the smallest angle.

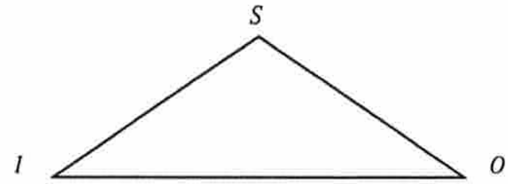
In an **equilateral triangle**, all the sides are the same length, and all the angles measure 60 degrees each.

In the equilateral triangle ΔEQI below, all the sides are of equal length, and all the angles are 60 degrees.



In an **isosceles triangle**, two of the three sides are the same length, and two of the three angles are the same size as each other.

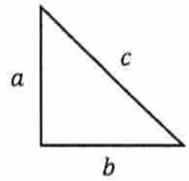
In the isosceles triangle ΔISO below, side $\overline{IS}$ is the same length as side $\overline{SO}$. Also, $\angle SIO$ and $\angle SOI$ have the same degree measurement as each other.



A **right triangle** is a triangle that includes a ninety-degree angle as one of its three angles.

A special relationship exists between the measurements of the sides of a right triangle: If you take the lengths of the two shorter sides and square them, and then add those two squares together, the resulting amount is the square of the length of the longest side.

In the right triangle below, $a^2 + b^2 = c^2$.



The expression of this relationship, $a^2 + b^2 = c^2$, is called the "**Pythagorean Theorem.**"

A "**Pythagorean triple**" is a set of three numbers that can all be the lengths of the sides of the same right triangle. Memorizing four of these sets will make your life easier on the SAT.

$\{3, 4, 5\}$ is a Pythagorean triple because $3^2 + 4^2 = 5^2$

$\{1, 1, \sqrt{2}\}$ is a Pythagorean triple because $1^2 + 1^2 = \sqrt{2}^2$

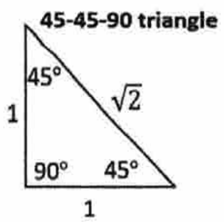
$\{1, \sqrt{3}, 2\}$ is a Pythagorean triple because $1^2 + \sqrt{3}^2 = 2^2$

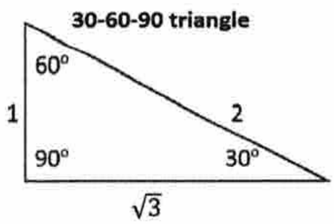
$\{5, 12, 13\}$ is a Pythagorean triple because $5^2 + 12^2 = 13^2$

When we multiply each number in a Pythagorean triple by the same number, we get another Pythagorean triple.

If we know $\{3, 4, 5\}$ is a Pythagorean triple, then we also know $\{6, 8, 10\}$ is a Pythagorean triple, because $\{6, 8, 10\}$ is what we get when we multiply every number in $\{3, 4, 5\}$ by 2.

In a $\{1, 1, \sqrt{2}\}$ right triangle, the angle measurements are $45°, 45°, 90°$.
In a $\{1, \sqrt{3}, 2\}$ right triangle, the angle measurements are $30°, 60°, 90°$.

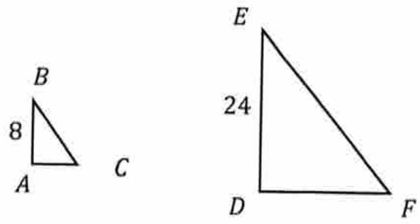




Two triangles are **similar triangles** if they have all the same angle measurements.

Between two similar triangles, the relationship between any two corresponding sides is the same as between any other two corresponding sides.

Triangles ΔABC and ΔDEF below are similar. Side $\overline{AB}$ has length 8, and side $\overline{DE}$ has length 24, so every side measurement in ΔDEF must be three times the corresponding side in ΔABC.



The formula for the **area of a triangle** is given in the front of every real SAT Math section. That formula is $A = \frac{1}{2}bh$, which means that the area of a triangle is equal to one half the length of the base of the triangle multiplied by the height of the triangle.

In every triangle, the length of each side must be less than the sum of the lengths of the other sides. (Otherwise, the triangle wouldn't be able to "close," because the longest side would be too long for the other two sides to touch.)

You can see in this diagram that the longest side is longer than the two shorter sides combined, which means the two shorter sides are too far apart to connect and "close" the triangle.

Points and Lines

A unique **line** can be drawn to connect any two **points.**

Between any two points on a line, there is a **midpoint** that is halfway between the two points.

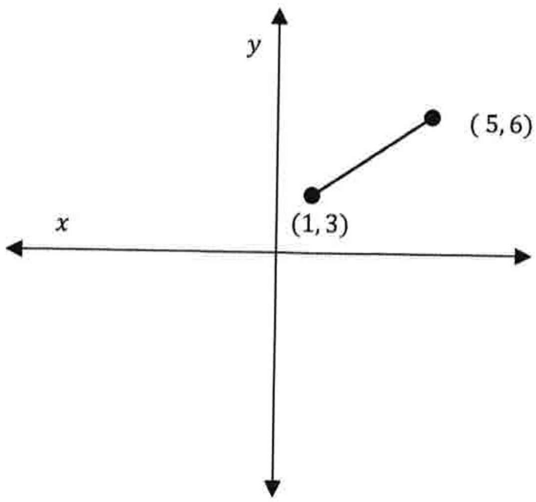
Any three or more points may or may not fall on the same line. If they do, we say the points are **collinear.**

We can find the **distance between any two points** on the xy-coordinate plane as long as we know the coordinates of each point. We can use the distance formula to find the distance between the point (x_1, y_1) and (x_2, y_2). That formula looks like this:

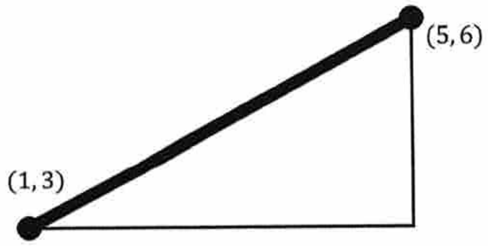
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We can also avoid memorizing a formula by just understanding the distance between the two points as the hypotenuse of a right triangle. We can then determine the lengths of the two sides of that right triangle, and use them to find the length of the hypotenuse. (This is ultimately what the distance formula does, but I find that test-takers tend to make more mistakes trying to memorize a complicated formula than they do just sketching out the right triangle as we'll see in the example.)

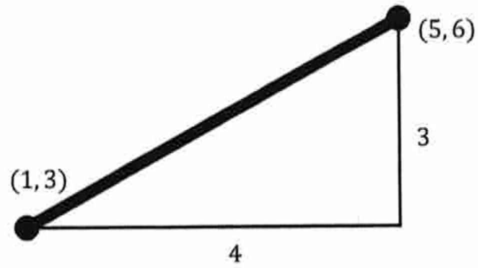
Imagine that we're trying to find the distance between $(1, 3)$ and $(5, 6)$.



Let's zoom in on that line segment, and draw in the sides of the triangle.



The vertical distance between the two points is the difference between their y-coordinates, which is $6 - 3$, or 3. This is the length of the vertical leg. The horizontal distance between the two points is the difference between their x-coordinates, which is $5 - 1$, or 4. This is the length of the horizontal leg. Let's add those lengths to the diagram.



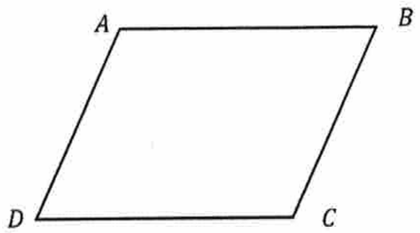
Now let's plug these numbers into the Pythagorean theorem to find the length of the hypotenuse.

$a^2 + b^2 = c^2$	(Pythagorean theorem)
$4^2 + 3^2 = c^2$	(plug in the two side lengths)
$16 + 9 = c^2$	(square the two side lengths)
$25 = c^2$	(simplify)
$5 = c$	(take square root of both sides)

So the distance between $(1, 3)$ and $(5, 6)$ is 5.

Parallelograms

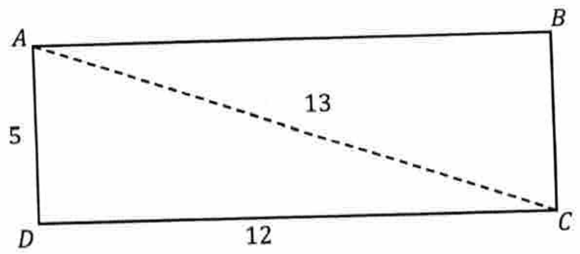
A parallelogram is a four-sided figure where both pairs of opposite sides are parallel to each other.
In a parallelogram, opposite angles are equal to each other, and the measures of all the angles added up together equal 360.
In ▱ $ABCD$ below, all the interior angles taken together equal 360°, and opposite angles have equal measurements.



Rectangles

Rectangles are special parallelograms where all the angles measure 90°. In a rectangle, if you know the lengths of the sides then you can always figure out the length from one corner to the opposite corner by using the Pythagorean theorem.

In the rectangle below, all angles are right angles, and we can use the Pythagorean theorem to determine that the diagonal $\overline{AC}$ must have a length of 13, since $5^2 + 12^2 = 13^2$.



Squares

Squares are special rectangles where all the sides have equal length.

Area

The **area** of a two-dimensional figure is the amount of two-dimensional space that the figure covers.

Area is always measured in square units.

All the area formulas you need for the SAT appear in the beginning of each Math section, so there's no need to memorize them—you just need to know how to use them.

Perimeters (Squares, Rectangles, Circles)

The **perimeter** of a two-dimensional object is the sum of the lengths of its sides, or, for a circle, the distance around the circle.

To find the perimeter of a non-circle, just add up the lengths of its sides.

The perimeter of a circle is called the circumference—see the section on circles for more information.

Other Polygons

The SAT might give you questions about special polygons, like pentagons, hexagons, octagons, and so on.

The sum of the angle measurements of any polygon can be determined with a simple formula: Where s is the number of sides of the polygon, the sum of the angle measurements is $(s-2) \times 180$.

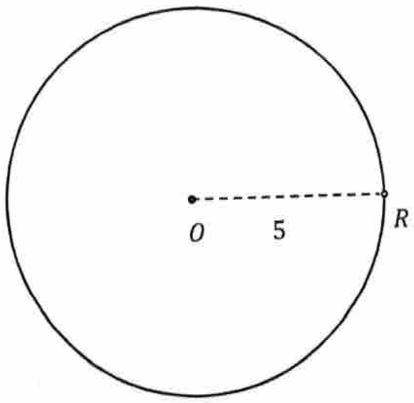
> A triangle has 3 sides, so the sum of its angle measurements is given by $(3-2) \times 180$, which is the same thing as 1×180, which is the same thing as 180. So the angles in a triangle add up to 180 degrees.
>
> A hexagon has 6 sides, so the sum of its angle measurements is $(6-2) \times 180$, or 4×180, which is 720. So all the angles in a hexagon add up to 720 degrees.

To find the perimeter of any polygon, just add up the lengths of the sides.

To find the area of a polygon besides a triangle, parallelogram, or circle, just divide the polygon into smaller triangles, polygons, and/or circles and find the areas of these pieces. A real SAT math question involving this concept will always lend itself to this solution nicely.

Circles

A **circle** is the set of points in a particular plane that are all equidistant from a single point, called the **center**.

> Circle O has a center at point O, and consists of all the points in one plane that are 5 units from the center:
>
> 
>

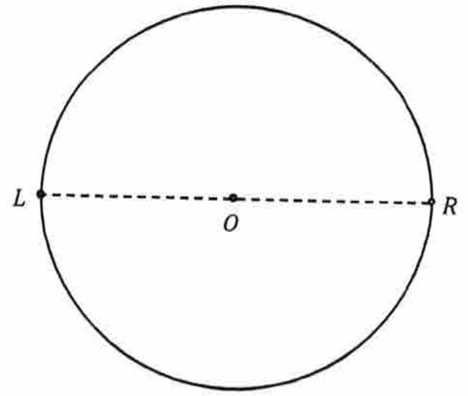
A **radius** is a line segment drawn from the center point of a circle to the edge of the circle.

> In circle O above, $\overline{OR}$ is a radius because it runs from the center of the circle (O) to the edge of the circle at point R.

All the radii of a circle have the same length, since all the points on the circle are the same distance from the center point.

A **diameter** is a line segment drawn from one edge of a circle, through the center of the circle, all the way to the opposite edge.

$\overline{LR}$ is a diameter of circle O because it starts at one edge of the circle, stretches through the center of the circle, and stops at the opposite edge of the circle.



Because a diameter can be broken into two opposite radii, a diameter always has a length equal to twice the radius of the circle.

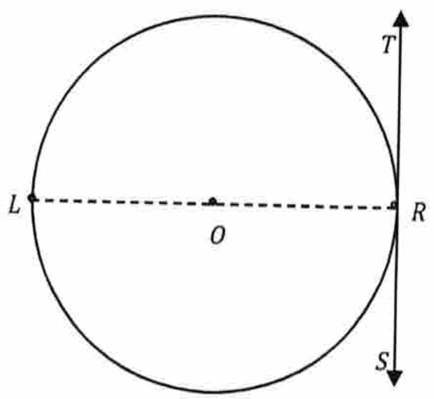
A diameter of a circle is the longest line segment that can be drawn through the circle.

The perimeter of a circle is called the **circumference.**

The formulas for area of a circle ($A = \pi r^2$) and circumference of a circle ($C = 2\pi r$) appear in the beginning of all real SAT Math sections, so there's no need to memorize them if you don't already know them. (In each formula, r represents the radius of the circle.)

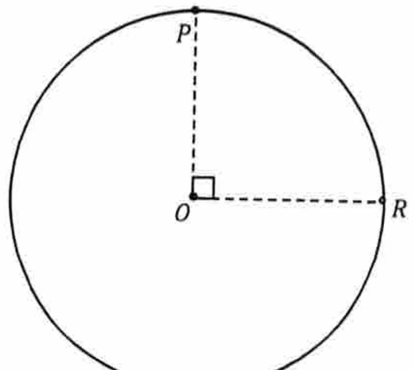
A **tangent** is a line that intersects a circle at only one point. A tangent is perpendicular to the radius ending at the shared point.

Circle O has a tangent line $\overline{TS}$ that intersects the circle at point R, and is perpendicular to radius OR.



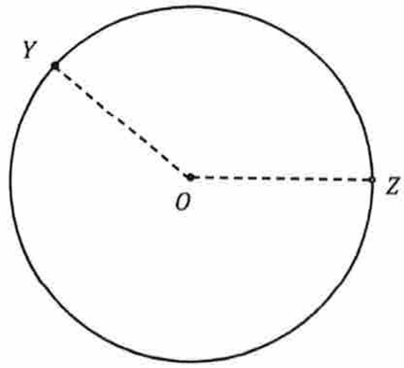
An **arc** is a portion of a circle that is measured in degrees, like an angle. We can measure an arc by drawing radii to the endpoints of the arc, and then measuring the angle formed by the radii at the center of the circle.

Circle O has a 90° arc $\widehat{PR}$, whose measure we can find by measuring the angle formed by radius PO and radius RO.



Two points on the circumference of a circle can define two different arcs. One arc is the shorter distance between the two points, while the other arc is the longer distance between the two points. The context in the problem often makes it clear which arc is relevant to the question. When the context isn't sufficient, we can use the term **minor arc** to refer to the arc that covers the shorter distance between the two points, and **major arc** to refer to the arc that covers the longer distance between the two points.

Minor arc $\widehat{YZ}$ measures 135°. Major arc $\widehat{YZ}$ measures 225°.

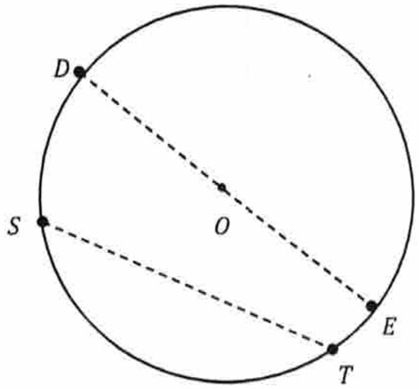


A **central angle** is an angle whose vertex is the center of a circle and whose sides are radii of that circle

In the previous figure, $\angle YOZ$ is a central angle whose measure is 135°.

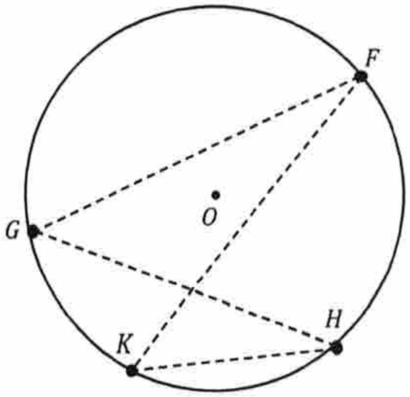
A **chord** is a line segment whose ends are points on the circumference of a circle. Note that a diameter is a chord that passes through the center of the circle.

$\overline{ST}$ is a chord on circle O. $\overline{DE}$ is a diameter of circle O, because $\overline{DE}$ is a chord that passes through the center of circle O.



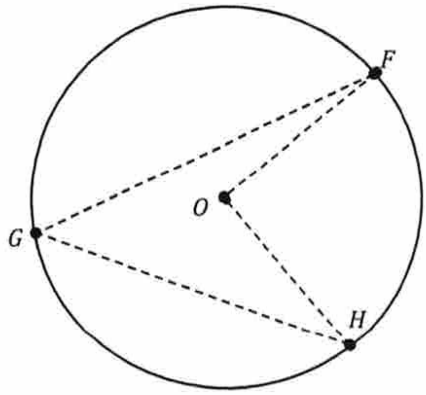
An **inscribed angle** is an angle created by two chords that meet on one end to form a vertex. We can "move" the vertex of an inscribed angle anywhere along the circumference the circle, and the measure of the inscribed angle will remain unchanged as long as the other endpoints of the angle don't move.

$\angle FGH$ is an inscribed angle in circle O. $\angle FKH$ is another inscribed angle in circle O. $\angle FGH$ and $\angle FKH$ have the same degree measure because they are inscribed angles of the same circle, and they share the endpoints F and H.



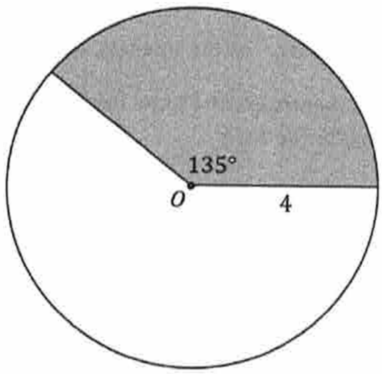
The degree measure of an inscribed angle is half the degree measure of a central angle with the same endpoints.

The angle measure of $\angle FGH$ is half the angle measure of $\angle FOH$ because $\angle FGH$ is an inscribed angle in circle O and $\angle FOH$ is a central angle in circle O and they share the endpoints F and H.



A **sector** is a portion of a circle defined by the center of the circle and two points on the circle, kind of like a slice of pizza. We can find the area of a sector by finding the area of the entire circle, and then multiplying by the fraction of the circle represented by the sector.

Let's find the area of the shaded sector. Circle O has a radius of 4, so the area of circle O is πr^2, or $\pi(4)^2$, or 16π. The shaded sector has a central angle of $135°$, so the sector represents $\frac{135°}{360°}$ of the total area of the circle. That means the area of the sector is $16\pi \times \frac{135°}{360°}$, or 6π.

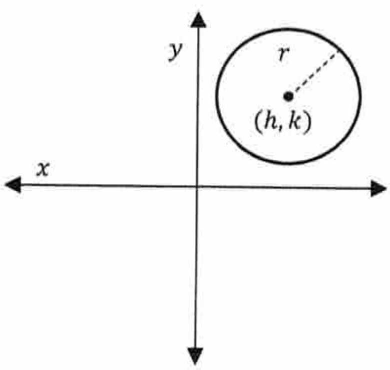


The **standard equation for a circle** with a center at the origin and a radius of r is $x^2 + y^2 = r^2$

The above equation is for a circle with a center at the origin. A circle whose center isn't at the origin can be expressed like this:

$$(x-h)^2 + (y-k)^2 = r^2$$

The circle described by the above equation will have a center at (h, k).



Sometimes, to get an equation for a circle into the above form, we'll need to complete the square. See the section on completing the square under "Polynomials" earlier in this Toolbox on page 181 for more information.

Solid Geometry

On the SAT, solid geometry may involve cubes, rectangular solids, prisms, cylinders, cones, spheres, or pyramids.

All necessary volume formulas will be given to you, so there's no need to memorize them.

The surface area of a solid is the sum of the areas of its faces (except for spheres or other "rounded" solids, whose surface areas won't be tested on the SAT unless a question provides a formula for finding their surface areas).

Statistics

The **mean** or **arithmetic mean** of a set of numbers is the result you get when you add all the numbers together and then divide by the number of things that you added.

> The average of $\{4, 9, 92\}$ is 35, because $\frac{4+9+92}{3} = 35$.

The **median** of a set of numbers is the number that appears in the middle of the set when all the numbers in the set are arranged from least to greatest.

> The median of $\{4, 9, 92\}$ is 9, because when we arrange the three numbers from least to greatest, 9 is in the middle.

If there is an even number of elements in the set, then the median of that set is the arithmetic mean of the two numbers in the middle of the set when the elements of the set are arranged from least to greatest.

> The median of $\{4, 9, 11, 92\}$ is 10, because the number of elements in the set is even, and 10 is the average of the two numbers in the middle of the set (which are 9 and 11).

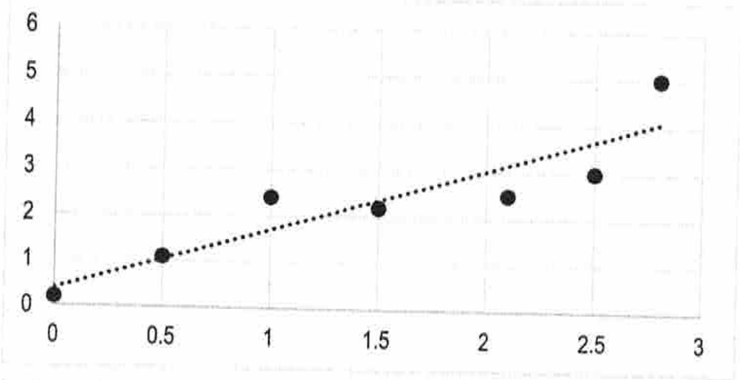
The **mode** of a set of numbers is the number that appears most frequently in the set.

> The mode of $\{7, 7, 23, 44\}$ is 7, because 7 appears more often than any other number in the set.

The **range** of a set of numbers is the difference between the highest number in the set and the lowest number in the set.

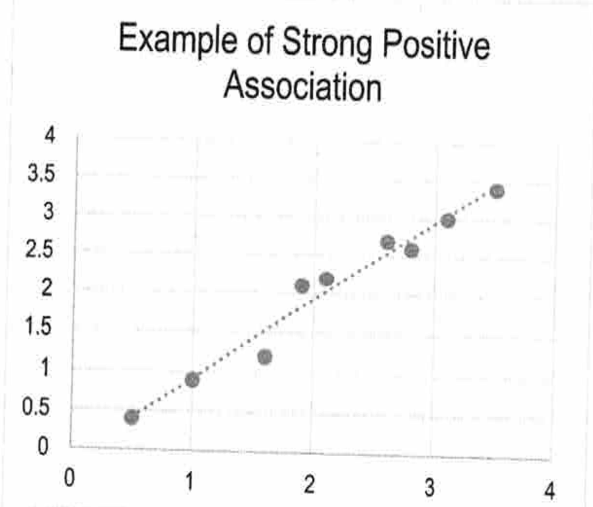
> The range of $\{7, 7, 23, 44\}$ is 37: 44 is the largest number in the set, 7 is the smallest, and $44 - 7 = 37$.

On a graph that shows a set of data points, the **line of best fit** is a line that demonstrates the trend in the data. In the sample figure below, the points represent actual data, while the dashed line demonstrates the *trend* in the data:

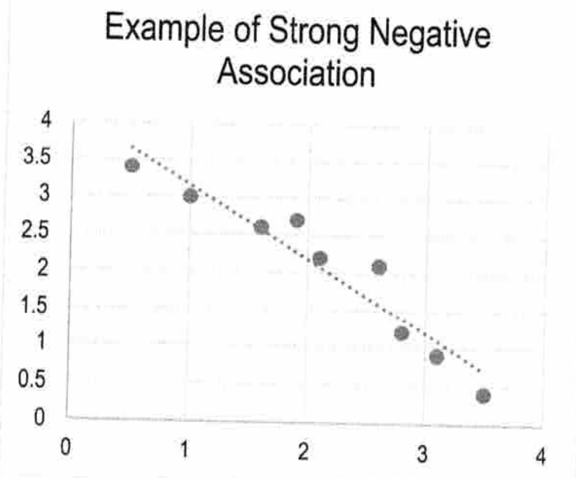


On the SAT, you'll never have to calculate a line of best fit, but you will need to understand that the line of best fit shows us the overall trend in the data—it doesn't represent *actual* data.

The closer the slope of a line of best fit is to 1, the stronger the **positive association** is between the x and y variables.



The closer the slope of a line of best fit is to -1, the stronger the **negative association** is between the x and y variables.



An **outlier** is a data point that doesn't follow the trend established by the other data points in the set. In a set of numbers, an outlier can simply be a number that is significantly greater than or less than the other numbers in the set.

> In the set {4,5,5,7,7,8,9,31}, 31 is an outlier.

Probability

The **probability** of an event is a fraction from 0 to 1 that describes how likely the event is to happen. If the fraction is closer to 1, the event is more likely to happen; if the fraction is closer to 0, the event is less likely to happen.

To determine the fraction, you first calculate the total number of possible outcomes and place this number in the denominator of the fraction; then, you determine the number of outcomes that satisfy the event's requirements, and place this number in the numerator of the fraction.

> The probability of rolling a 3 on a normal 6-sided die is $\frac{1}{6}$. There are 6 possible outcomes, so 6 goes in the denominator of the fraction. Out of those 6 outcomes, we only want one—the one where a 3 comes up—so 1 goes in the numerator.
>
> The probability of rolling an odd number on a normal 6-sided die is $\frac{3}{6}$. Again, there are 6 possible numbers we might roll, so 6 is our denominator. But now, since we want any odd number, the numbers 1, 3, and 5 all satisfy the requirements of our event, so there are 3 possible outcomes that we'll be happy with—that means 3 goes in the numerator. (Notice that we can reduce the probability to $\frac{1}{2}$.)

Probability fractions can be manipulated just like any other fractions.

To find the probability of two or more events happening in a sequence, we just find the probability of each event by itself, and then multiply them by each other.

> The probability of rolling double-sixes on two normal 6-sided dice is $\frac{1}{36}$, because the probability of rolling a six on either die is $\frac{1}{6}$, and $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

Trigonometry

The trigonometry that you need on the SAT is relatively basic and limited. The most important things you need to know are the three basic trigonometric ratios.

You've probably learned the three basic ratios in math class with the acronym "SOHCAHTOA."

"SOH" stands for Sine = Opposite / Hypotenuse

"CAH" stands for Cosine = Adjacent / Hypotenuse

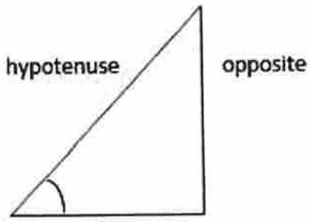
"TOA" stands for Tangent = Opposite / Adjacent

Given a right triangle:

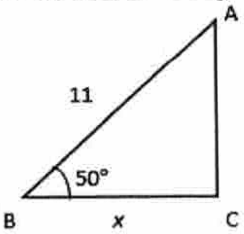
The **hypotenuse** is the side that's opposite the right angle.

The **opposite** side is the side across from the angle whose sine, cosine, or tangent we're evaluating.

The **adjacent** side is the side that's next to the angle we're evaluating (the one that isn't the hypotenuse).



You can use these ratios to solve for the lengths of sides or the measure of angles.



In the figure above, we're given the measure of an angle and the length of the hypotenuse. We can set up an expression for x, the adjacent side length, because we know that the cosine of an angle is equal to the length of the adjacent side divided by the length of the hypotenuse.

In this example, $\cos 50° = \frac{x}{11}$, so that means $11(\cos 50°) = x$.

In the context of trigonometry, you may see angles measured in degrees or radians. **Radians** are just another way to measure the size of an angle, and are usually expressed using π. $360° = 2\pi$ radians.

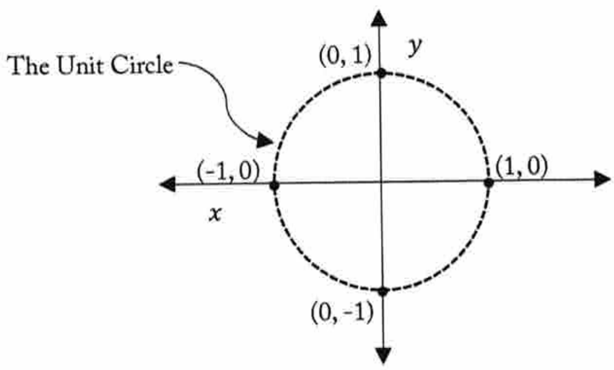
When you use your calculator to evaluate trigonometric expressions, always make sure your calculator is in the correct mode! If the expression uses degrees, your calculator should be in degree mode. If the expression uses radians, your calculator should be in radian mode.

The Unit Circle

The unit circle is a mathematical construct at the heart of trigonometry. It has many interesting applications in mathematical theory, but we're not concerned with those applications for the purposes of the SAT Math section. In fact, questions on the SAT don't require us to know anything about the unit circle—still, it sometimes happens that understanding the unit circle will allow us to answer a question more quickly. For these reasons, our discussion of the unit circle will be limited to the following:

1. giving a formal definition of the unit circle
2. explaining how the various trig functions can be evaluated for a given angle on the unit circle
3. explaining the difference between measuring angles in radians and degrees

The unit circle is a circle drawn in the xy-coordinate plane with its center at (0, 0) and a radius of 1 unit. The unit circle therefore passes through the points (1, 0), (0, 1), (-1, 0), and (0, -1):

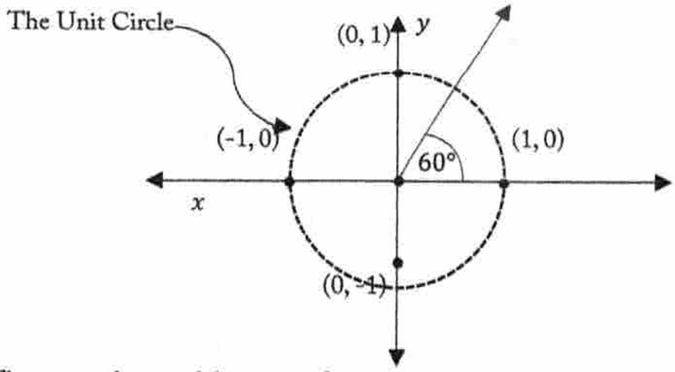


We can use the unit circle to determine the trig values for a given angle by taking the following steps.

1. Lay the angle over the unit circle in the following way:
   a. the vertex of the angle goes on the point (0, 0)
   b. one leg of the angle extends along the positive portion of the x-axis.
   c. the angle's other leg opens upward, counter-clockwise. If the angle is greater than 180°, then it opens past the point (-1, 0) and towards the point (0, -1).
2. Note the point where the leg from part (c) intercepts the unit circle:

- The cosine of the angle is the x-coordinate of the point in Step 2.
- The sine of the angle is the y-coordinate of the point in Step 2.
- All other trig values for the angle can be calculated from the sine and cosine.

Let's see how we could find the trig values for a 60° angle using the unit circle. To do this, we start by imagining that we're placing the angle over the unit circle in the way we just described:



The x-coordinate of the point of intersection is the angle's cosine, and the y-coordinate of the point of intersection is the angle's sine. As we can see, the sine of a 60° angle is approximately 0.866, and the cosine is 0.5. These two values allow us to determine the value of tangent for the angle as well:

$$\text{Tangent } 60° = (\text{sine } 60°/\text{cosine } 60°) \approx \frac{0.866}{0.5} \approx 1.732$$

Conclusion

We've just covered all the math concepts that the College Board will allow itself to cover on the SAT. As I mentioned at the beginning of the Toolbox, it's important to keep in mind that simply knowing these concepts is not enough to guarantee a good score on the SAT. It's much more important to focus on the design of the SAT Math section and learn to take apart challenging questions.